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THE EFFECT OF SMALL INTERNAL AND EXTERNAL DAMPING ON THE STABILITY OF DISTRIBUTED NON-CONSERVATIVE SYSTEMS†

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The effect of small internal and external damping on the stability of distributed non-conservative systems is investigated. A theory is constructed for the qualitative and quantitative description of the "destabilization paradox" in these systems, one manifestation of which is an abrupt drop in the critical load and frequency when small dissipative forces are taken into account. The theory is base on an analysis of the bifurcations of multiple eigenvalues of non-self-adjoint differential operators that depend on parameters. Explicit formulae are obtained for the collapse of multiple eigenvalues with Keldysh chains of arbitrary length, for linear differential operators that depend analytically on a complex spectral parameter and are smooth functions of a vector of real parameters. It is shown that the "destabilization paradox" is related to the perturbation by small damping of a double eigenvalue of a circulatory system with a Keldysh chain of length 2, which is responsible for the formation of a singularity on the boundary of the stability domain. Formulae describing the behaviour of the eigenvalues of a non-conservative system when the load and dissipation parameters are varied are described. Explicit expressions are obtained for the jumps in the critical loads and frequency of the loss of stability. Approximations are obtained in analytical form of the asymptotic stability domain in the parameter space of the system. The *stabilization effect*, in which a distributed circularity system is stabilized by small dissipative forces and which consists of an increase in the critical load, is explained, and stabilization conditions are derived. As a mechanical example, the stability of a visco-elastic rod with small internal and external damping is investigated; unlike earlier publications, it is shown that the boundary of the stability domain has a "Whitney umbrella" singularity. The dependence of the critical load on the internal and external friction parameters is obtained in analytical form, yielding an explicit expression for the jump in critical load. On the basis of the analytical relations, the domains of stabilization and destabilization in the parameter space of the system are constructed. It is shown that the analytical formulae are in good agreement with the numerical results of earlier research.

1. INTRODUCTION

Ziegler [1], when investigating the stability of a double pendulum subject to a follower load, reached the unexpected conclusion that the critical force of loss stability of a non-conservative system with negligibly small damping is significantly less than in the case when it is assumed from the very start that there is no damping in the system. This phenomenon, now known was the *destabilization paradox*, was subsequently observed in many non-conservative mechanical systems, both discrete and distributed [2–20]. Despite the large number of publications, the problems arising from the destabilization paradox still await a general solution, although, in Bolotin's position [2], they are of the greatest theoretical interest in non-conservative problems of stability.

As an illustration of the destabilization paradox, let us consider the transverse vibrations of a cantilever rod of a visco-elastic Kelvin–Voigt material, subject at its free end to a shear follower force q [2, 7]. In dimensionless variables, the equation of small vibrations of the rod and the boundary conditions are

$$\frac{\partial^4 y}{\partial x^4}+q\frac{\partial^2 y}{\partial x^2}+\eta\frac{\partial^5 y}{\partial x^4\partial t}+\mu\frac{\partial y}{\partial t}+\frac{\partial^2 y}{\partial t^2} = 0 \tag{1.1}$$

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$$y(0,t) = \frac{\partial y}{\partial x}(0,t) = 0, \quad \frac{\partial^2 y}{\partial x^2}(1,t) + \eta\frac{\partial^3 y}{\partial x^2 \partial t}(1,t) = \frac{\partial^3 y}{\partial x^3}(1,t) + \eta\frac{\partial^4 y}{\partial x^3 \partial t}(1,t) = 0 \tag{1.2}$$

The coefficient of internal damping η characterizes the visco-elastic properties of the material; the coefficient of external damping μ represents the resistance of the medium.

Seeking a solution in the form $y(x, t) = u(x)\exp\lambda t$, we arrive at the eigenvalue problem

$$(1+\eta\lambda)u''''_{xxxx} + qu''_{xx} + (\lambda^2 + \mu\lambda)u = 0 \tag{1.3}$$

$$u(0) = u'_x(0) = 0, \quad u''_{xx}(1) = u'''_{xxx}(1) = 0 \tag{1.4}$$

where λ is an eigenvalue, $u(x)$ is an eigenfunction, and the prime denotes differentiation with respect to the subscripts – in this case with respect to the variable $x \in [0, 1]$.

The system described by Eqs (1.1) and (1.2) is asymptotically stable if all the eigenvalues λ of problem (1.3), (1.4) have negative real parts and unstable if there is at least one eigenvalue in the right half of the complex plane ($\operatorname{Re}\lambda > 0$). The critical load $q_{cr}(\eta, \mu)$ characterizing the transition from stability to instability is determined by the condition that the real part of one or more eigenvalues should vanish ($\operatorname{Re}\lambda = 0$).

If the damping parameters in Eqs (1.1)–(1.4) are equated to zero, a *circulatory system* results [2, 5]. A circulatory system is stable (not asymptotically) if all its eigenvalues are imaginary and semi-simple, that is, the algebraic multiplicity of each eigenvalue is identical with the number of its eigenfunctions. As the load parameter q is varied, the eigenvalues move along the imaginary axis and at some value of $q = q_0$ two of them merge into one double eigenvalue $i\omega_0$, which then splits into a pair of complex conjugate eigenvalues [2, 22]. To the double eigenvalue $i\omega_0$ there corresponds a Keldysh chain of length 2, consisting of an eigenfunction u_0 and the associated function u_1, which satisfy the following equations and boundary conditions [7]

$$u''''_{0xxxx} + q_0 u''_{0xx} - \omega_0^2 u_0 = 0, \quad u_0(0) = u'_{0x}(0) = 0, \quad u''_{0xx}(1) = u'''_{0xxx}(1) = 0 \tag{1.5}$$

$$u''''_{1xxxx} + q_0 u''_{1xx} - \omega_0^2 u_1 = -2i\omega_0 u_0, \quad u_1(0) = u'_{1x}(0) = 0, \quad u''_{1xx}(1) = u'''_{1xxx}(1) = 0 \tag{1.6}$$

This means that when $q = q_0$ only one eigenfunction corresponds to the algebraically double eigenvalue, which is expressed in the appearance of a secular term of the form $(u_1(x) + tu_0(x))e^{i\omega_0 t}$ in the general solution of the boundary-value problem (1.1), (1.2). Keldysh chains generalize the well-known concept of Jordan chains in linear algebra [23–30]. Thus, the existence of a double eigenvalue $i\omega_0$, $\omega_0 > 0$, with a Keldysh chain of length 2, in the spectrum of the unperturbed problem, on the assumption that all other eigenvalues are pure imaginary and simple, corresponds to the boundary between the domains of stability and flutter (oscillatory instability) [31, 32].

It is well-known that system (1.1), (1.2), considered without damping ($\eta = 0$, $\mu = 0$), is stable for $0 \le q < q_0 = 20.05$ [21], but if allowance is made for arbitrary small internal damping ($\eta \to +0$, $\mu = 0$) the stable interval shrinks to $0 \le q < q_{cr} = 10.94 < q_0$. Consequently, this problem exhibits the destabilization paradox: when allowance is made for arbitrarily small internal damping the critical load falls abruptly. At the same time, the critical frequency also falls abruptly, from $\omega_0 = 11.02$ to $\omega_{cr} = 5.40$ [4, 7]. These effects are shown in Fig. 1 for $\mu = 0$. External damping ($\mu > 0$) reduces the destabilizing effect of internal damping [7]. We note that previous researchers solved partial mechanical problems similar to that considered above, using numerical or semi-analytical methods.

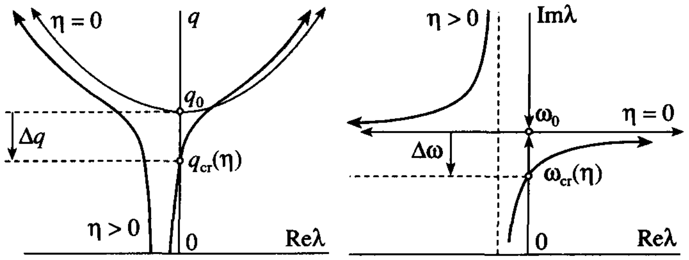

Fig. 1

The aim of the present paper is to develop analytical methods for investigating the spectrum of non-self-adjoint boundary-value problems for eigenvalues, depending on parameters, and use them to investigate the effect of small damping forces on the stability of distribute non-conservative system of a general type, including problem (1.1), (1.2) as a special case.

2. BIFURCATION OF MULTIPLE EIGENVALUES WITH KELDYSH CHAINS

Since the destabilization paradox proves to be related to the existence in the spectrum of an unperturbed circulatory system of a double eigenvalue with a Keldysh chain, a prerequisite for studying the paradox is a knowledge of the behaviour of multiple eigenvalues as functions of the parameters of the problem. To that end, we shall consider a generalized non-self-adjoint eigenvalue problem for a linear differential operator with boundary conditions [27, 30].

Let L denote a linear differential operator of order m with respect to the variable x, whose operation on a smooth function $u(x)$ is defined by

$$Lu = \sum_{j=0}^{m} l_j \frac{d^{m-j}u}{dx^{m-j}} \tag{2.1}$$

The coefficients $l_j(x, \lambda, \mathbf{p})$ of the operator L are smooth functions of the variable x; the function $l_0(x)$ is bounded below by a positive constant in the interval $x \in [0, 1]$. In addition, it is assumed that the coefficients $l_j(x, \lambda, \mathbf{p})$ are analytical functions of the complex spectral parameter λ and smooth functions of the real parameter vector $\mathbf{p} \in R^n$.

Let us call the matrix $\mathbf{U} = ||\mathbf{AB}||$ of order $m \times 2m$ and rank m, where each of the blocks $\mathbf{A}$ and $\mathbf{B}$ is of order $m \times m$, the matrix of boundary conditions. We define a vector $\mathbf{u} = (\mathbf{u}(0), \mathbf{u}(1))$ of dimension $2m$, where the components of the vectors

$$\mathbf{u}(\xi) = (u(\xi), u_x'(\xi), \ldots, u_x^{(m-1)}(\xi)), \quad \xi = 0, 1$$

are the values of the function $u(x)$ and its derivatives at the boundary point $x = 0$ and $x = 1$. Then

$$\mathbf{Uu} = \mathbf{Au}(0) + \mathbf{Bu}(1) \tag{2.2}$$

It is assumed that the elements of the matrices $\mathbf{A}(\lambda, \mathbf{p})$ and $\mathbf{B}(\lambda, \mathbf{p})$ are analytic functions of the complex spectral parameter λ and smooth functions of the vector of real parameters $\mathbf{p} \in R^n$.

In the interval $x \in [0, 1]$, consider the eigenvalue problem for the differential operator L with boundary conditions defined by the matrix $\mathbf{U}$

$$L(x, \lambda, \mathbf{p})u = 0, \quad \mathbf{U}(\lambda, \mathbf{p})\mathbf{u} = 0 \tag{2.3}$$

The boundary-value problem (2.3) has a non-trivial solution if and only if the characteristic determinant vanishes [26, 27, 30]:

$$\det(\mathbf{AY}(0) + \mathbf{BY}(1)) = 0 \tag{2.4}$$

where the elements of the matrix $\mathbf{Y}(x)$ are defined by the relations $Y_{ij}(x) = y_{jx}^{(i-1)}(x)$, $(i, j = 1, 2, \ldots, m)$, while $y_1(x), y_2(x), \ldots, y_m(x)$ is a fundamental system of solutions of the differential equation (2.3). For some fixed vector $\mathbf{p} = \mathbf{p}_0$, the eigenvalue λ_0 to which the eigenfunction u_0 corresponds is a root of the characteristic equation (2.4).

Multiplying Eq. (2.3) by the function $\bar{v}(x)$, where the bar denotes complex conjugation, and integrating by parts, we get

$$\int_0^1 \bar{v} L u dx = \int_0^1 u \overline{L^* v} dx + \bar{\mathbf{v}}^T(1)\mathbf{L}(1)\mathbf{u}(1) - \bar{\mathbf{v}}^T(0)\mathbf{L}(0)\mathbf{u}(0) \tag{2.5}$$

where [26]

$$L^*v = \sum_{j=0}^{m}(-1)^{m-j}\frac{d^{m-j}}{dx^{m-j}}(\overline{l_j(x)}v) \tag{2.6}$$

and the matrices $\mathbf{L}(0)$ and $\mathbf{L}(1)$ are the values at the points $x = 0$ and $x = 1$ of the matrix $\mathbf{L}(x)$ of order $m \times m$ whose elements $L_{ij}(x)$ are expressed as follows in terms of the coefficients of the differential operator l_j and their derivatives with respect to x:

$$L_{ij}(x) = \sum_{k=i-1}^{m-j}(-1)^k C_k^{i-1}\frac{d^{k-i+1}}{dx^{k-i+1}}l_{m-j-k}, \quad C_k^{i-1} = \begin{cases}\dfrac{k!}{(i-1)!(k-i+1)!}, & k \geq i \quad 1 \geq 0\\ 0, \quad k < i-1\end{cases} \tag{2.7}$$

The components of the vectors

$$\mathbf{v}(\xi) = (v(\xi), v_x'(\xi), \ldots, v_x^{(m-1)}(\xi)), \quad \xi = 0, 1$$

are the values of the function $v(x)$ and its derivatives at points $x = 0$ and $x = 1$. We put $\mathbf{v} = (\mathbf{v}(0), \mathbf{v}(1))$.

Let us consider a matrix $\tilde{\mathbf{U}} = \|\tilde{\mathbf{A}}\tilde{\mathbf{B}}\|$ of order $m \times m$, where the matrices $\tilde{\mathbf{A}}(\lambda, \mathbf{p})$ and $\tilde{\mathbf{B}}(\lambda, \mathbf{p})$ of order $m \times m$ may depend, generally speaking, on the spectral parameter λ and the real parameter vector $\mathbf{p}$. We choose matrices $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ such that the partitioned matrix of order $2m \times 2m$ constructed from the matrices $\mathbf{U}$, $\tilde{\mathbf{U}}$ is non-singular in the neighbourhood of the point $\mathbf{p} = \mathbf{p}_0$ and the eigenvalue $\lambda = \lambda_0$. Then

$$\begin{Vmatrix} -\mathbf{L}(0) & \mathbf{O} \\ \mathbf{O} & \mathbf{L}(1) \end{Vmatrix} = \mathbf{V}^*\tilde{\mathbf{U}} - \tilde{\mathbf{V}}^*\mathbf{U} \tag{2.8}$$

where the asterisk denotes Hermitian conjugation (in the case of matrices this is the operation of transportation and complex conjugation), $\mathbf{O}$ is the $m \times m$ zero matrix, and $\mathbf{V}$ and $\tilde{\mathbf{V}}$ are the $m \times 2m$ matrices defined by

$$\begin{Vmatrix} -\tilde{\mathbf{V}} \\ \mathbf{V} \end{Vmatrix}^* = \begin{Vmatrix} -\mathbf{L}(0) & \mathbf{O} \\ \mathbf{O} & \mathbf{L}(1) \end{Vmatrix} \begin{Vmatrix} \mathbf{A} & \mathbf{B} \\ \tilde{\mathbf{A}} & \tilde{\mathbf{B}} \end{Vmatrix}^{-1} \tag{2.9}$$

Differentiation of Eq. (2.8) with respect to λ gives

$$\begin{Vmatrix} -\mathbf{L}'_\lambda(0) & \mathbf{O} \\ \mathbf{O} & \mathbf{L}'_\lambda(1) \end{Vmatrix} = (\mathbf{V}'_{\bar{\lambda}})^*\tilde{\mathbf{U}} + \mathbf{V}^*\tilde{\mathbf{U}}'_\lambda - (\tilde{\mathbf{V}}'_{\bar{\lambda}})^*\mathbf{U} - \tilde{\mathbf{V}}^*\mathbf{U}'_\lambda \tag{2.10}$$

where the prime denotes differentiation with respect to the spectral parameter λ or $\bar{\lambda}$ (the bar denotes complex conjugation).

Taking relation (2.8) into consideration in Eq. (2.5), we obtain Lagrange's formula for the operator L [26].

$$(Lu, v) - (u, L^*v) = (\mathbf{V}\mathbf{v})^*\tilde{\mathbf{U}}\mathbf{u} - (\tilde{\mathbf{V}}\mathbf{v})^*\mathbf{U}\mathbf{u}, \quad (u, v) = \int_0^1 u(x)\bar{v}(x)dx \tag{2.11}$$

where (u, v) is the Hermitian scalar product of the functions u and v.

If it is assumed that the matrices $\tilde{\mathbf{B}}$ and $\mathbf{S} = \mathbf{A} - \mathbf{B}\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{A}}$ are non-singular, then Schur's formula [33]

$$\det\begin{Vmatrix} \mathbf{A} & \mathbf{B} \\ \tilde{\mathbf{A}} & \tilde{\mathbf{B}} \end{Vmatrix} = \det\tilde{\mathbf{B}}\,\det(\mathbf{A} - \mathbf{B}\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{A}}) \neq 0 \tag{2.12}$$

and the matrices $\mathbf{V}$ and $\tilde{\mathbf{V}}$ of order $m \times 2m$ can be written explicitly as

$$\mathbf{V} = \|(\mathbf{L}(0)\mathbf{S}^{-1}\mathbf{B}\tilde{\mathbf{B}}^{-1})^* \; (\mathbf{L}(1)\tilde{\mathbf{B}}^{-1} + \mathbf{L}(1)\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{A}}\mathbf{S}^{-1}\mathbf{B}\tilde{\mathbf{B}}^{-1})^*\| \tag{2.13}$$

$$\tilde{\mathbf{V}} = \|(\mathbf{L}(0)\mathbf{S}^{-1})^* \; (\mathbf{L}(1)\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{A}}\mathbf{S}^{-1})^*\| \tag{2.14}$$

The eigenvalue problem for the operator L^*

$$L^*(\bar{\lambda}, \mathbf{p})\upsilon = 0, \quad \mathbf{V}(\bar{\lambda}, \mathbf{p})\mathbf{v} = 0 \tag{2.15}$$

is the adjoint of problem (2.3), and the operator L^* defined by Eqs (2.6) is the adjoint of the operator L (2.1). Adjoint operators L and L^*, with the corresponding boundary conditions (the second equalities of (2.3) and (2.15)), satisfy the relation $(Lu, \upsilon) = (u, L^*\upsilon)$ [26].

Let us assume that the spectrum of problem (2.3) is discrete at the point $\mathbf{p}_0$ and in its neighbourhood. Put $L_0 = L(\lambda_0, \mathbf{p}_0)$, $\mathbf{U}_0 = \mathbf{U}(\lambda_0, \mathbf{p}_0)$, and consider the following smooth curve in the n-dimensional parameter space depending on the real parameter $\epsilon \geq 0$

$$\mathbf{p}(\epsilon) = \mathbf{p}_0 + \epsilon\dot{\mathbf{p}} + \frac{\epsilon^2}{2}\ddot{\mathbf{p}} + o(\epsilon^2) \tag{2.16}$$

where the dot denotes differentiation with respect to ε and the derivatives are evaluated at $\epsilon = 0$. With this perturbation, the operator $L(\lambda, \mathbf{p}(\epsilon))$ can be represented as a series

$$L(\lambda, \mathbf{p}(\epsilon)) = \sum_{r=0}^{\infty} \frac{(\lambda - \lambda_0)^r}{r!}\left(\frac{\partial^r L}{\partial \lambda^r} + \epsilon\frac{\partial^r L_1}{\partial \lambda^r} + \epsilon^2\frac{\partial^r L_2}{\partial \lambda^r} + o(\epsilon^2)\right) \tag{2.17}$$

$$\frac{\partial^r L_1}{\partial \lambda^r} = \sum_{j=1}^{n} \frac{\partial^{r+1} L}{\partial \lambda^r \partial p_j}\dot{p}_j, \quad \frac{\partial^r L_2}{\partial \lambda^r} = \frac{1}{2}\sum_{j=1}^{n} \frac{\partial^{r+1} L}{\partial \lambda^r \partial p_j}\ddot{p}_j + \frac{1}{2}\sum_{j,t=1}^{n} \frac{\partial^{r+2} L}{\partial \lambda^r \partial p_j \partial p_t}\dot{p}_j\dot{p}_t \tag{2.18}$$

when $r = 0$, formulae (2.18) yield expressions for the operators L_1 and L_2. Accordingly, the matrix of boundary conditions $\mathbf{U}(\lambda, \mathbf{p}(\epsilon))$ becomes

$$\mathbf{U}(\lambda, \mathbf{p}(\epsilon)) = \sum_{r=0}^{\infty} \frac{(\lambda - \lambda_0)^r}{r!}\left(\frac{\partial^r \mathbf{U}}{\partial \lambda^r} + \epsilon\frac{\partial^r \mathbf{U}_1}{\partial \lambda^r} + \epsilon^2\frac{\partial^r \mathbf{U}_2}{\partial \lambda^r} + o(\epsilon^2)\right) \tag{2.19}$$

$$\frac{\partial^r \mathbf{U}_1}{\partial \lambda^r} = \sum_{j=1}^{n} \frac{\partial^{r+1} \mathbf{U}}{\partial \lambda^r \partial p_j}\dot{p}_j, \quad \frac{\partial^r \mathbf{U}_2}{\partial \lambda^r} = \frac{1}{2}\sum_{j=1}^{n} \frac{\partial^{r+1} \mathbf{U}}{\partial \lambda^r \partial p_j}\ddot{p}_j + \frac{1}{2}\sum_{j,t=1}^{n} \frac{\partial^{r+2} \mathbf{U}}{\partial \lambda^r \partial p_j \partial p_t}\dot{p}_j\dot{p}_t \tag{2.20}$$

where the partial derivatives are evaluated at $\mathbf{p} = \mathbf{p}_0$, $\lambda = \lambda_0$. When $r = 0$, formulae (2.20) yield expressions for the matrices $\mathbf{U}_1$ and $\mathbf{U}_2$.

The simple eigenvalue λ_0. Let us assume that the eigenvalue λ_0 at the point $\mathbf{p} = \mathbf{p}_0$ is a simple root of Eq. (2.4) with eigenfunction u_0. The eigenfunction u_0 satisfies the following equation with boundary conditions

$$L_0 u_0 = 0, \quad \mathbf{U}_0 \mathbf{u}_0 = 0 \tag{2.21}$$

and the eigenfunction υ_0 of the complex-conjugate eigenvalue $\bar{\lambda}_0$ of the adjoint operator is a solution of the eigenvalue problem

$$L_0^* \upsilon_0 = 0, \quad \mathbf{V}_0 \mathbf{v}_0 = 0 \tag{2.22}$$

Then the perturbed eigenvalue $\lambda(\epsilon)$ and the eigenfunction $u(\epsilon)$ are represented as series in ϵ [34]

$$\lambda = \lambda_0 + \lambda_1\epsilon + \lambda_2\epsilon^2 + \ldots, \quad u = u_0 + w_1\epsilon + w_2\epsilon^2 + \ldots \tag{2.23}$$

Put

$$\mathbf{w}_j = (\mathbf{w}_j(0), \mathbf{w}_j(1)), \quad \mathbf{w}_j(\xi) = (w_j(\xi), w'_{jx}(\xi), \ldots, w_{jx}^{(m-1)}(\xi)), \quad \xi = 0, 1, \quad j = 1, 2, \ldots$$

Substituting expansions (2.17)–(2.20) and (2.23) into Eqs (2.3) and collecting the coefficients of ϵ, we obtain equations and boundary conditions that must be satisfied by the function w_1 in the first expansion (2.23)

$$L_0 w_1 = -L_1 u_0 - \lambda_1 L'_\lambda u_0, \quad \mathbf{U}_0 \mathbf{w}_1 = -\mathbf{U}_1 \mathbf{u}_0 - \lambda_1 \mathbf{U}'_\lambda \mathbf{u}_0 \tag{2.24}$$

Multiplying both sides of Eq. (2.4) scalarly by the function v_0 and using the Lagrange formula (2.11), which may be written, using the equation and boundary condition (2.22), as

$$(L_0 w_1, v_0) = \mathbf{v}_0^* \tilde{\mathbf{V}}_0^* \mathbf{U}_1 \mathbf{u}_0 + \lambda_1 \mathbf{v}_0^* \tilde{\mathbf{V}}_0^* \mathbf{U}'_\lambda \mathbf{u}_0 \tag{2.25}$$

we find the coefficient λ_1 in the first expansion of (2.23)

$$\lambda_1 = -\frac{(L_1 u_0, v_0) + \mathbf{v}_0^* \tilde{\mathbf{V}}_0^* \mathbf{U}_1 \mathbf{u}_0}{(L'_\lambda u_0, v_0) + \mathbf{v}_0^* \tilde{\mathbf{V}}_0^* \mathbf{U}'_\lambda \mathbf{u}_0} \tag{2.26}$$

The double eigenvalue λ_0: *the regular case*. We will now consider the case of a double eigenvalue λ_0 with Keldysh chain of length 2, consisting of an eigenfunction u_0 and associated function u_1 satisfying the following equations and boundary conditions [26]

$$L_0 u_0 = 0, \quad \mathbf{U}_0 \mathbf{u}_0 = 0 \tag{2.27}$$

$$L_0 u_1 = -L'_\lambda u_0, \quad \mathbf{U}_0 \mathbf{u}_1 = -\mathbf{U}'_\lambda \mathbf{u}_0 \tag{2.28}$$

We multiply Eq. (2.27) scalarly by the function v_1 and Eq. (2.28) by the function v_0, integrate the resulting expressions by parts using formulae (2.5) for the operators L_0 and L_λ, and then add. The result is the relation

$$\begin{gathered}(u_0, L_0^* v_1 + L_{\bar\lambda}^{*\prime} v_0) + (u_1, L_0^* v_0) + \\ + \mathbf{v}_1^* \begin{Vmatrix} -\mathbf{L}_0(0) & \mathbf{O} \\ \mathbf{O} & \mathbf{L}_0(1) \end{Vmatrix} \mathbf{u}_0 + \mathbf{v}_0^* \begin{Vmatrix} -\mathbf{L}_0(0) & \mathbf{O} \\ \mathbf{O} & \mathbf{L}_0(1) \end{Vmatrix} \mathbf{u}_1 + \mathbf{v}_0^* \begin{Vmatrix} -\mathbf{L}'_\lambda(0) & \mathbf{O} \\ \mathbf{O} & \mathbf{L}'_\lambda(1) \end{Vmatrix} \mathbf{u}_0 = 0\end{gathered} \tag{2.29}$$

Taking relations (2.8) and (2.10) into account, Eq. (2.29) transforms to

$$\begin{gathered}(u_0, L_0^* v_1 + L_{\bar\lambda}^{*\prime} v_0) + (u_1, L_0^* v_0) + (\mathbf{V}_0 \mathbf{v}_1 + \mathbf{V}'_{\bar\lambda} \mathbf{v}_0)^* \tilde{\mathbf{U}}_0 \mathbf{u}_0 - (\tilde{\mathbf{V}}_0 \mathbf{v}_1 + \tilde{\mathbf{V}}'_{\bar\lambda} \mathbf{v}_0)^* \mathbf{U}_0 \mathbf{u}_0 + \\ + (\mathbf{V}_0 \mathbf{v}_0)^* (\tilde{\mathbf{U}}_0 \mathbf{u}_1 + \tilde{\mathbf{U}}'_\lambda \mathbf{u}_0) - (\tilde{\mathbf{V}}_0 \mathbf{v}_0)^* (\mathbf{U}_0 \mathbf{u}_1 + \mathbf{U}'_\lambda \mathbf{u}_0) = 0\end{gathered} \tag{2.30}$$

The functions v_0 and v_1, which satisfy the equations and boundary conditions

$$L_0^* v_0 = 0, \quad \mathbf{V}_0 \mathbf{v}_0 = 0 \tag{2.31}$$

$$L_0^* v_1 = -L_{\bar\lambda}^{*\prime} v_0, \quad \mathbf{V}_0 \mathbf{v}_1 = -\mathbf{V}'_{\bar\lambda} \mathbf{v}_0 \tag{2.32}$$

from the *adjoint* Keldysh chain of the double eigenvalue λ_0. The equations of the adjoint chains (2.27), (2.28) and (2.31), (2.32) have the same form and make Eq. (2.30) an identity.

Multiplying Eq. (2.28) scalarly by the function v_0 and using the Lagrange identity (2.11), with due note of the equation and boundary conditions (2.28) of the form

$$(L_0 u_1, v_0) = \mathbf{v}_0^* \tilde{\mathbf{V}}_0^* \mathbf{U}'_\lambda \mathbf{u}_0 \tag{2.33}$$

we obtain the orthogonality condition

$$(L'_\lambda u_0, v_0) + \mathbf{v}_0^* \tilde{\mathbf{V}}_0^* \mathbf{U}'_\lambda \mathbf{u}_0 = 0 \tag{2.34}$$

Equation (2.34) is a relation between the eigenfunctions of the adjoint problems in the case of a double eigenvalue with Keldysh chain of length 2. However, it is also valid for a multiple eigenvalue with Keldysh chain of arbitrary length. For circulatory non-conservative systems the orthogonality condition (2.34) characterizes the onset (boundary) of flutter [35, 37].

The perturbed double eigenvalue $\lambda(\epsilon)$ and its eigenfunction $u(\epsilon)$ are represented by Newton–Puiseaux series [34]

$$\lambda = \lambda_0 + \lambda_1 \epsilon^{1/2} + \lambda_2 \epsilon + \lambda_3 \epsilon^{3/2} + \lambda_4 \epsilon^2 + \dots \tag{2.35}$$

$$u = u_0 + w_1 \epsilon^{1/2} + w_2 \epsilon + w_3 \epsilon^{3/2} + w_4 \epsilon^2 + \dots \tag{2.36}$$

As before, we now substitute expansions (2.17)–(2.20) and (2.35), (2.36) into the eigenvalue problem (2.3) and collect coefficients of like powers of the small parameter ϵ. Hence we find that the functions w_1, w_2 and w_3 satisfy the following equations and boundary conditions

$$L_0 w_1 = -\lambda_1 L'_\lambda u_0, \quad \mathbf{U}_0 \mathbf{w}_1 = -\lambda_1 \mathbf{U}'_\lambda \mathbf{u}_0 \tag{2.37}$$

$$\begin{aligned} L_0 w_2 &= -\lambda_1 L'_\lambda w_1 - \lambda_2 L'_\lambda u_0 - L_1 u_0 - \frac{\lambda_1^2}{2!} L''_{\lambda\lambda} u_0 \\ \mathbf{U}_0 \mathbf{w}_2 &= -\lambda_1 \mathbf{U}'_\lambda \mathbf{w}_1 - \lambda_2 \mathbf{U}'_\lambda \mathbf{u}_0 - \mathbf{U}_1 \mathbf{u}_0 - \frac{\lambda_1^2}{2!} \mathbf{U}''_{\lambda\lambda} \mathbf{u}_0 \end{aligned} \tag{2.38}$$

$$\begin{aligned} L_0 w_3 &= -\lambda_1 L'_\lambda w_2 - \left(L_1 + \lambda_2 L'_\lambda + \frac{\lambda_1^2}{2!} L''_{\lambda\lambda}\right) w_1 - \left(\lambda_1 L'_{1\lambda} + \lambda_3 L'_\lambda + \lambda_1 \lambda_2 L''_{\lambda\lambda} + \frac{\lambda_1^3}{3!} L'''_{\lambda\lambda\lambda}\right) u_0 \\ \mathbf{U}_0 \mathbf{w}_3 &= -\lambda_1 \mathbf{U}'_\lambda \mathbf{w}_2 - \left(\mathbf{U}_1 + \lambda_2 \mathbf{U}'_\lambda + \frac{\lambda_1^2}{2!} \mathbf{U}''_{\lambda\lambda}\right) \mathbf{w}_1 - \left(\lambda_1 \mathbf{U}'_{1\lambda} + \lambda_3 \mathbf{U}'_\lambda + \lambda_1 \lambda_2 \mathbf{U}''_{\lambda\lambda} + \frac{\lambda_1^3}{3!} \mathbf{U}'''_{\lambda\lambda\lambda}\right) \mathbf{u}_0 \end{aligned} \tag{2.39}$$

Comparing Eqs (2.37) and (2.38), we find that the structure of the function w_1 is

$$w_1 = \lambda_1 u_1 + \gamma u_0, \tag{2.40}$$

where γ is an arbitrary coefficient. Evaluating the scalar product of Eq. (2.38) and the function v_0, substituting expression (2.40) for the function w_1 into the result and using Eqs (2.31) and (2.32) and the Lagrange identity (2.11), we obtain the coefficient λ_1 in expansion (2.35)

$$\lambda_1^2 = -\frac{1}{\sigma_2}((L_1 u_0, v_0) + \mathbf{v}_0^* \tilde{\mathbf{V}}_0^* \mathbf{U}_1 \mathbf{u}_0), \quad \sigma_2 = \sum_{r=1}^{2} \frac{1}{r!}((L_\lambda^{(r)} u_{2-r}, v_0) + \mathbf{v}_0^* \tilde{\mathbf{V}}_0^* \mathbf{U}_\lambda^{(r)} \mathbf{u}_{2-r}) \tag{2.41}$$

To find the next expansion coefficient λ_2, we evaluate the scalar product of Eq. (2.39) and the function v_0 and then use Lagrange's formula (2.11). We obtain

$$\begin{aligned} &\gamma \frac{\lambda_1^2}{2}((L''_{\lambda\lambda} u_0, v_0) + (\tilde{\mathbf{V}}_0 \mathbf{v} L_0)^* \mathbf{U}''_{\lambda\lambda} \mathbf{u}_0) + \lambda_1((L'_\lambda w_2, v_0) + (\tilde{\mathbf{V}}_0 \mathbf{v} L_0)^* \mathbf{U}'_\lambda \mathbf{w}_2) + \\ &+ \gamma((L_1 u_0, v_0) + (\tilde{\mathbf{V}}_0 \mathbf{v}_0)^* \mathbf{U}_1 \mathbf{u}_0) + \lambda_1((L_1 u_1 + L'_{1\lambda} u_0, v_0) + (\tilde{\mathbf{V}}_0 \mathbf{v}_0)^* (\mathbf{U}_1 \mathbf{u}_1 + \mathbf{U}'_{1\lambda} \mathbf{u}_0)) + \\ &+ \lambda_1^3\left(\left(\frac{1}{2!} L''_{\lambda\lambda} u_1 + \frac{1}{3!} L'''_{\lambda\lambda\lambda} u_0, v_0\right) + (\tilde{\mathbf{V}}_0 \mathbf{v}_0)^* \left(\frac{1}{2!} \mathbf{U}''_{\lambda\lambda} \mathbf{u}_1 + \frac{1}{3!} L'''_{\lambda\lambda\lambda} \mathbf{u}_0\right)\right) + \\ &+ \lambda_1 \lambda_2((L'_\lambda u_1 + L''_{\lambda\lambda} u_0, v_0) + (\tilde{\mathbf{V}}_0 \mathbf{v}_0)^* (\mathbf{U}'_\lambda \mathbf{u}_1 + \mathbf{U}''_{\lambda\lambda} \mathbf{u}_0)) = 0 \end{aligned} \tag{2.42}$$

On the other hand, the scalar product of Eq. (2.38) and the function v_1, using the integration formula (2.5) and the identities (2.8) and (2.10), yields the relation

$$(L'_\lambda w_2, v_0) + (\tilde{\mathbf{V}}_0 \mathbf{v} L_0)^* \mathbf{U}'_\lambda \mathbf{w}_2 = -(\tilde{\mathbf{V}}_0 \mathbf{v}_1 + \tilde{\mathbf{V}}'_\lambda \mathbf{v}_0)^* \mathbf{U}_0 \mathbf{w}_2 + \\ + \lambda_1^2 (L'_\lambda u_1, v_1) + \gamma \lambda_1 (L'_\lambda u_0, v_1) + (L_1 u_0, v_1) + \lambda_2 (L'_\lambda u_0, v_1) + \frac{\lambda_1^2}{2} (L''_{\lambda\lambda} u_0, v_1) \tag{2.43}$$

In addition, we have the identity

$$(L'_\lambda u_0, v_1) + (\tilde{\mathbf{V}}_0 \mathbf{v}_1 + \tilde{\mathbf{V}}'_\lambda \mathbf{v}_0)^* \mathbf{U}'_\lambda \mathbf{u}_0 = (L'_\lambda u_1, v_0) + (\tilde{\mathbf{V}}_0 \mathbf{v}_0)^* \mathbf{U}'_\lambda \mathbf{u}_1 \tag{2.44}$$

as follows from Eqs (2.27), (2.28) and (2.31), (2.32), as well as (2.10).

Using relations (2.41), (2.43) and (2.44), we deduce from Eq. (2.42) that

$$\lambda_2 = -\frac{1}{2\sigma_2}((L_1 u_0, v_1) + (L_1 u_1, v_0) + (L'_{1\lambda} u_0, v_0)) - \\ -\frac{1}{2\sigma_2}(\mathbf{v}_1^* \tilde{\mathbf{V}}_0^* \mathbf{U}_1 \mathbf{u}_0 + \mathbf{v}_0^* \tilde{\mathbf{V}}_0^* \mathbf{U}_1 \mathbf{u}_1 + \mathbf{v}_0^* (\tilde{\mathbf{V}}^* \mathbf{U}_1)'_\lambda \mathbf{u}_0 + \lambda_1^2 Q) \tag{2.45}$$

where

$$Q = (L'_\lambda u_1, v_1) + \frac{1}{2!}(L''_{\lambda\lambda} u_0, v_1) + \frac{1}{2!}(L''_{\lambda\lambda} u_1, v_0) + \frac{1}{3!}(L'''_{\lambda\lambda\lambda} u_0, v_0) + \\ + (\tilde{\mathbf{V}}_0 \mathbf{v}_1 + \tilde{\mathbf{V}}'_\lambda \mathbf{v}_0)^* \left(\mathbf{U}'_\lambda \mathbf{u}_1 + \frac{1}{2!} \mathbf{U}''_{\lambda\lambda} \mathbf{u}_0 \right) + (\tilde{\mathbf{V}}_0 \mathbf{v}_0)^* \left(\frac{1}{2!} \mathbf{U}''_{\lambda\lambda} \mathbf{u}_1 + \frac{1}{3!} \mathbf{U}'''_{\lambda\lambda\lambda} \mathbf{u}_0 \right) \tag{2.46}$$

and the number σ_2 is defined by the second relation of (2.41).

The double eigenvalue λ_0: *the degenerate case*. The expansions (2.35) with coefficients defined by Eqs (2.41) and (2.45) hold provided $\lambda_1 \neq 0$. The case $\lambda_1 = 0$, or, equivalently,

$$(L_1 u_0, v_0) + \mathbf{v}_0^* \mathbf{V}_0^* \mathbf{U}_1 \mathbf{u}_0 = 0 \tag{2.47}$$

is *degenerate* and needs special consideration. Substitution of the expansions (2.31) and (2.32) together with (2.17)–(2.20) into the eigenvalue problem (2.3) with the condition $\lambda_1 = 0$ leads to the following equations and boundary conditions

$$L_0 w_1 = 0, \quad \mathbf{U}_0 \mathbf{w}_1 = 0 \tag{2.48}$$

$$L_0 w_2 = -\lambda_2 L'_\lambda u_0 - L_1 u_0, \quad \mathbf{U}_0 \mathbf{w}_2 = -\lambda_2 \mathbf{U}'_\lambda \mathbf{u}_0 - \mathbf{U}_1 \mathbf{u}_0 \tag{2.49}$$

$$L_0 w_4 = -\lambda_3 L'_\lambda w_1 - \lambda_2 L'_\lambda w_2 - L_1 w_2 - \lambda_2 L'_{1\lambda} u_0 - \lambda_2^2 \frac{1}{2} L''_{\lambda\lambda} u_0 - \lambda_4 L'_\lambda u_0 - L_2 u_0 \tag{2.50}$$

$$\mathbf{U}_0 \mathbf{w}_4 = -\lambda_3 \mathbf{U}'_\lambda \mathbf{w}_1 - \lambda_2 \mathbf{U}'_\lambda \mathbf{w}_2 - \mathbf{U}_1 \mathbf{w}_2 - \lambda_2 \mathbf{U}'_{1\lambda} \mathbf{u}_0 - \lambda_2^2 \frac{1}{2} \mathbf{U}''_{\lambda\lambda} \mathbf{u}_0 - \lambda_4 \mathbf{U}'_\lambda \mathbf{u}_0 - \mathbf{U}_2 \mathbf{u}_0$$

Solving Eqs (2.48) and (2.49), we obtain

$$w_1 = \beta u_0, \quad w_2 = \lambda_2 u_1 + \gamma u_0 + \hat{w}_2 \tag{2.51}$$

where β and γ are arbitrary constants, the function $\hat{w}_2$ is a solution of the boundary-value problem

$$L_0 \hat{w}_2 = -L_1 u_0, \quad \mathbf{U}_0 \hat{\mathbf{w}}_2 = -\mathbf{U}_1 \mathbf{u}_0 \tag{2.52}$$

and $\hat{\mathbf{w}}_2 = (\hat{w}_2(0), \hat{w}'_{2x}(0), \dots, \hat{w}_{2x}^{(m-1)}(0), \hat{w}_2(1), \hat{w}'_{2x}(1), \dots, \hat{w}_{2x}^{(m-1)}(1))$.

Multiplying Eq. (2.49) scalarly by the function v_1 and using Lagrange's formula (2.11) and the expressions (2.8) and (2.10), we obtain

$$(L'_\lambda w_2, v_0) + (\tilde{\mathbf{V}}_0\mathbf{v}_0)^*\mathbf{U}'_\lambda\mathbf{w}_2 = -(\tilde{\mathbf{V}}_0\mathbf{v}_1 + \tilde{\mathbf{V}}'_\lambda\mathbf{v}_0)^*\mathbf{U}_0\mathbf{w}_2 + \lambda_2(L'_\lambda u_0, v_1) + (L_1 u_0, v_1) \quad (2.53)$$

In addition, the scalar product of Eq. (2.50) and the function v_0, using Lagrange's formula (2.11), the boundary conditions (2.50) and expression (2.15), gives

$$\begin{aligned}&\lambda_2((L'_\lambda w_2, v_0) + (\tilde{\mathbf{V}}_0\mathbf{v}_0)^*\mathbf{U}'_\lambda\mathbf{w}_2) + (L_1 w_2, v_0) + (\tilde{\mathbf{V}}_0\mathbf{v}_0)^*\mathbf{U}_1\mathbf{w}_2 + (L_2 u_0, v_0) + (\tilde{\mathbf{V}}_0\mathbf{v}_0)^*\mathbf{U}_2\mathbf{u}_0 + \\ &+\lambda_2((L'_{1\lambda}u_0, v_0) + (\tilde{\mathbf{V}}_0\mathbf{v}_0)^*\mathbf{U}'_{1\lambda}\mathbf{u}_0) + \frac{1}{2}\lambda_2^2((L''_{\lambda\lambda}u_0, v_0) + (\tilde{\mathbf{V}}_0\mathbf{v}_0)^*\mathbf{U}''_{\lambda\lambda}\mathbf{u}_0) = 0\end{aligned} \quad (2.54)$$

After substituting (2.53) into Eq. (2.54) and using the identity (2.44) and the boundary conditions (2.38), this equation becomes

$$\begin{aligned}&\lambda_2^2\sigma_2 + \lambda_2(L_1 u_0, v_1) + (L_1 u_1, v_0) + (L'_{1\lambda}u_0, v_0) + \mathbf{v}_1^*\tilde{\mathbf{V}}_0^*\mathbf{U}_1\mathbf{u}_0 + \mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\mathbf{U}_1\mathbf{u}_1 + \\ &+\mathbf{v}_0^*(\tilde{\mathbf{V}}^*\mathbf{U}_1)'_\lambda\mathbf{u}_0) + (L_2 u_0, v_0) + (L_1\hat{w}_2, v_0) + (\tilde{\mathbf{V}}_0\mathbf{v}_0)^*(\mathbf{U}_2\mathbf{u}_0 + \mathbf{U}_1\hat{\mathbf{w}}_2) = 0\end{aligned} \quad (2.55)$$

The quantity σ_0 is defined by the second relation of (2.41).

Thus, the expansion $\lambda = \lambda_0 + \epsilon\lambda_2 + o(\epsilon)$ and the quadratic equation (2.55) describe the collapse of the double eigenvalue in the degenerate case (2.47).

In the case when the boundary conditions do not depend on the parameters or the operator L is a matrix, formula (2.55) is simplified [19]

$$\lambda_2^2 + \lambda_2\frac{(L_1u_0, v_1) + (L_1u_1, v_0) + (L'_{1\lambda}u_0, v_0)}{(L'_\lambda u_1, v_0) + 1/2(L''_\lambda u_0, v_0)} + \frac{(L_2u_0, v_0) + (L_1\hat{w}_2, v_0)}{(L'_\lambda u_1, v_0) + 1/2(L''_\lambda u_0, v_0)} = 0 \quad (2.56)$$

But if the operator L has the form $Lu \equiv l(\mathbf{p})u - \lambda u$, where $l(\mathbf{p})$ is a linear differential operator with constant coefficients and the boundary conditions do not depend on λ, then formula (2.55) takes the form obtained in [32].

An eigenvalue of arbitrary multiplicity. We will now introduce a formula describing the collapse of a μ-tuple eigenvalue λ_0 with Keldysh chain of length μ, consisting of an eigenfunction u_0 and associated eigenfunctions $u_1, \ldots, u_{\mu-1}$. The functions forming the Keldysh chain satisfy the following equations with boundary conditions [23–25]

$$\begin{aligned}&L_0u_0 = 0, \quad \mathbf{U}_0\mathbf{u}_0 = 0\\ &L_0u_j = -\sum_{r=1}^{j}\frac{1}{r!}L_\lambda^{(r)}u_{j-r}, \quad \mathbf{U}_0\mathbf{u}_j = -\sum_{r=1}^{j}\frac{1}{r!}\mathbf{U}_\lambda^{(r)}\mathbf{u}_{j-r}, \quad j = 1, \ldots, \mu-1\end{aligned} \quad (2.57)$$

where the partial derivatives are evaluated at $\lambda = \lambda_0$ and $\mathbf{p} = \mathbf{p}_0$. The Keldysh chain for the complex-conjugate eigenvalue $\bar{\lambda}_0$ of the operator L_0^*, the Hermitian conjugate of L_0, satisfies the equations

$$\begin{aligned}&L_0^*v_0 = 0, \quad \mathbf{V}_0\mathbf{v}_0 = 0\\ &L_0^*v_j = -\sum_{r=1}^{j}\frac{1}{r!}L_{\bar{\lambda}}^{*(r)}v_{j-r}, \quad \mathbf{V}_0\mathbf{v}_j = -\sum_{r=1}^{j}\frac{1}{r!}\mathbf{V}_{\bar{\lambda}}^{(r)}\mathbf{v}_{j-r}, \quad j = 1, \ldots, \mu-1\end{aligned} \quad (2.58)$$

Multiplying Eq. (2.57) scalarly by the function v_0 and using the Lagrange identity (2.11), we arrive at the orthogonality relations

$$\sum_{r=1}^{j}\frac{1}{r!}((L_\lambda^{(r)}u_{j-r}, v_0) + \mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\mathbf{U}_\lambda^{(r)}\mathbf{u}_{j-r}) = 0, \quad j = 1, \ldots, \mu-1 \quad (2.59)$$

Equations (2.59) include the orthogonality condition (2.34).

Let us consider a smooth variation of the parameter vector (2.16). The perturbed eigenvalue $\lambda(\epsilon)$ and eigenfunction $u(\epsilon)$ are represented by Newton–Puiseaux series [34]

$$\lambda = \lambda_0 + \lambda_1\epsilon^{1/\mu} + \lambda_2\epsilon^{2/\mu} + \ldots + \lambda_{\mu-1}\epsilon^{(\mu-1)/\mu} + \lambda_\mu\epsilon + \ldots \tag{2.60}$$

$$u = u_0 + w_1\epsilon^{1/\mu} + w_2\epsilon^{2/\mu} + \ldots + w_{\mu-1}\epsilon^{(\mu-1)/\mu} + w_\mu\epsilon + \ldots \tag{2.61}$$

the substitute expansions (2.60), (2.61) together with (2.17)–(2.20) into the eigenvalue problem (2.3) and collect coefficients of like powers of the small parameter ε. The first $\mu - 1$ equations with boundary conditions become

$$L_0 w_r = -\sum_{j=0}^{r-1}\left(\sum_{\sigma=1}^{r-j}\frac{1}{\sigma!}L_\lambda^{(\sigma)}\sum_{|\alpha|_\sigma = r-j}\lambda_{\alpha_1}\ldots\lambda_{\alpha_\sigma}\right)w_j, \quad r = 1, \ldots, \mu-1 \tag{2.62}$$

$$\mathbf{U}_0\mathbf{w}_r = -\sum_{j=0}^{r-1}\sum_{\sigma=1}^{r-j}\left(\sum_{|\alpha|_\sigma = r-j}\lambda_{\alpha_1}\ldots\lambda_{\alpha_\sigma}\right)\frac{1}{\sigma!}\mathbf{U}_\lambda^{(\sigma)}\mathbf{w}_j, \quad |\alpha|_\sigma = \alpha_1 + \ldots + \alpha_\sigma \tag{2.63}$$

where $w_0 = u_0$ and the subscripts $\alpha_1, \ldots, \alpha_{\mu-1}$ are positive integers. The equation and boundary conditions for the function w_μ have the form

$$L_0 w_\mu = -L_1 w_0 - \sum_{j=0}^{\mu-1}\left(\sum_{\sigma=1}^{\mu-j}\frac{1}{\sigma!}L_\lambda^{(\sigma)}\sum_{|\alpha|_\sigma = \mu-j}\lambda_{\alpha_1}\ldots\lambda_{\alpha_\sigma}\right)w_j \tag{2.64}$$

$$\mathbf{U}_0\mathbf{w}_\mu = -\mathbf{U}_1\mathbf{w}_0 - \sum_{j=0}^{\mu-1}\sum_{\sigma=1}^{\mu-j}\left(\sum_{|\alpha|_\sigma = \mu-j}\lambda_{\alpha_1}\ldots\lambda_{\alpha_\sigma}\right)\frac{1}{\sigma!}\mathbf{U}_\lambda^{(\sigma)}\mathbf{w}_j \tag{2.65}$$

Comparison of Eqs (2.62) and (2.63) with the equations of the Keldysh chain (2.58) yields the coefficients w_r in the expansions (2.61)

$$w_r = \sum_{j=1}^{r} u_j \sum_{|\alpha|_j = r}\lambda_{\alpha_1}\ldots\lambda_{\alpha_j}, \quad r = 1, \ldots, \mu-1 \tag{2.66}$$

which satisfy the boundary conditions (2.63). Using the functions (2.66), we transform Eqs (2.64) and (2.65) to the form

$$L_0 w_\mu = -L_1 u_0 - \lambda_1^\mu\sum_{r=1}^{\mu}\frac{1}{r!}L_\lambda^{(r)}u_{\mu-r} + \sum_{j=1}^{\mu-1}L_0 u_j\sum_{|\alpha|_j = \mu}\lambda_{\alpha_1}\ldots\lambda_{\alpha_j} \tag{2.67}$$

$$\mathbf{U}_0\mathbf{w}_\mu = -\mathbf{U}_1\mathbf{u}_0 - \lambda_1^\mu\sum_{r=1}^{\mu}\frac{1}{r!}\mathbf{U}_\lambda^{(r)}\mathbf{u}_{\mu-r} + \sum_{j=1}^{\mu-1}\mathbf{U}_0\mathbf{u}_j\sum_{|\alpha|_j = \mu}\lambda_{\alpha_1}\ldots\lambda_{\alpha_j} \tag{2.68}$$

Multiplying Eq. (2.67) scalarly by v_0, and using the fact that

$$(L_0 u_j, v_0) + \mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\mathbf{U}_0\mathbf{u}_j = 0, \quad j = 1, \ldots, \mu-1 \tag{2.69}$$

as well as the Lagrange identity (2.11), which here has the form

$$(L_0 w_\mu, v_0) = \mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\mathbf{U}_1\mathbf{u}_0 + \lambda_1^\mu\sum_{r=1}^{\mu}\frac{1}{r!}\mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\mathbf{U}_\lambda^{(r)}\mathbf{u}_{\mu-r} - \sum_{j=1}^{\mu-1}\mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\mathbf{U}_0\mathbf{u}_j\sum_{|\alpha|_j = \mu}\lambda_{\alpha_1}\ldots\lambda_{\alpha_j} \tag{2.70}$$

we find the coefficient λ_1 in the expansions (2.60):

$$\lambda_1^{\mu} = -\frac{1}{\sigma_{\mu}}((L_1 u_0, \upsilon_0) + \mathbf{v}_0^* \tilde{\mathbf{V}}_0^* \mathbf{U}_1 \mathbf{u}_0), \quad \sigma_{\mu} = \sum_{r=1}^{\mu} \frac{1}{r!}((L_{\lambda}^{(r)} u_{\mu-r}, \upsilon_0) + \mathbf{v}_0^* \tilde{\mathbf{V}}_0^* \mathbf{U}_{\lambda}^{(r)} \mathbf{u}_{\mu-r}) \tag{2.71}$$

Thus, we have obtained an explicit function describing the collapse of multiple eigenvalues with Keldysh chain of arbitrary length, for differential operators that are analytic functions of the complex spectral parameter and smooth functions of the real parameter vector.

3. ANALYTICAL DESCRIPTION OF THE "DESTABILIZATION PARADOX"

We will now formulate a general eigenvalue problem that arises in stability analysis for viscoelastic systems

$$L(\lambda, q, \mathbf{k})u \equiv N(q)u + \lambda D(\mathbf{k})u + \lambda^2 M u = 0 \tag{3.1}$$

$$\mathbf{U}(q, \mathbf{k}, \lambda)\mathbf{u} \equiv \mathbf{U}_N(q)\mathbf{u} = 0 \tag{3.2}$$

The coefficients of the differential operators N, D and M of order m and the matrix $\mathbf{U}_N$ of order $m \times 2m$ are assumed to be real. The operator $N(q)$ and the matrix $\mathbf{U}_N(q)$ are smooth functions of the real load parameter $q \geq 0$, and the coefficients of the differential operator $D(\mathbf{k})$, which is of the order of at most m, are smooth functions of the vector of real dissipation parameters $\mathbf{k} = (k_1, \ldots, k_{n-1})$; it is assumed that if $\mathbf{k} = 0$, then $D(0) = 0$. It is also assumed that the operator M is independent of the parameter. Thus, the perturbation of the system by small dissipative forces ($|\mathbf{k}| \ll 1$) is regular [34].

It is assumed that the unperturbed system

$$N(q)u + \lambda^2 M u = 0, \quad \mathbf{U}_N(q)\mathbf{u} = 0 \tag{3.3}$$

considered over an interval $0 \leq q < q_0$, has a discrete spectrum consisting of simple pure imaginary eigenvalues $\lambda = i\omega$ and is consequently stable; at $q = q_0$, however, there is a pair of double eigenvalues $\pm i\omega_0$, $\omega_0 > 0$ with a Keldysh chain of length 2 (instability) [31, 32]. The eigenfunction u_0 and the associated eigenfunction u_1 of the eigenvalue $i\omega_0$ satisfy Eqs (2.27) and (2.28), which are now

$$L_0 u_0 \equiv N(q_0)u_0 - \omega_0^2 M u_0 = 0, \quad \mathbf{U}_0 \mathbf{u}_0 \equiv \mathbf{U}_N(q_0)\mathbf{u}_0 = 0 \tag{3.4}$$

$$N(q_0)u_1 - \omega_0^2 M u_1 = -2i\omega_0 M u_0, \quad \mathbf{U}_N(q_0)\mathbf{u}_1 = 0 \tag{3.5}$$

Since the coefficients of the operators and matrices occurring in Eqs (3.4) and (3.5) are real, we may assume that the eigenvalue u_0 is real. Then the associated eigenfunction u_1 will be pure imaginary. All the other eigenvalues $\pm i\omega_{0,j}$, $\omega_{0,j} > 0$ of the unperturbed system at $q = q_0$ are assumed to be simple and pure imaginary. Consequently, at $q = q_0$, when there are no dissipative forces ($\mathbf{k} = 0$), the non-conservative system is on the boundary between the domains of stability and flutter [31, 32].

Since the coefficients of the operator L are polynomials in the spectral parameter λ, it follows that the matrix $\mathbf{L}(x)$ defined by formula (2.7) has the form

$$\mathbf{L}(x) = \lambda^2 \mathbf{M}(x) + \lambda \mathbf{D}(x, \mathbf{k}) + \mathbf{N}(x, q) \tag{3.6}$$

The components of the matrices $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{N}$ are found from formulae analogous to (2.7); they consist of the coefficients of the operators M, D and N and their derivatives with respect to x, respectively. Choosing a real matrix $\tilde{\mathbf{U}}$ of order $m \times 2m$, we use formula (2.9) to find matrices $\mathbf{V}$ and $\tilde{\mathbf{V}}$ defining the boundary conditions of the adjoint eigenvalue problem.

The eigenfunction υ_0 and associated eigenfunction υ_1 of the complex-conjugate eigenvalue $-i\omega_0$ satisfy the following equations and boundary conditions

$$N^*(q_0)\upsilon_0 - \omega_0^2 M^* \upsilon_0 = 0, \quad \mathbf{V}_0 \mathbf{v}_0 = 0 \tag{3.7}$$

$$N^*(q_0)v_1 - \omega_0^2 M^* v_1 = 2i\omega_0 M^* v_0, \quad \mathbf{V}_0\mathbf{v}_1 = -\frac{\partial \mathbf{V}}{\partial \lambda}\mathbf{v}_0 \tag{3.8}$$

Since the matrices $\mathbf{U}_0$ and $\tilde{\mathbf{U}}_0$ are real and the matrix polynomial $\mathbf{L}(x)$ defined by Eq. (3.6) has real coefficients, it follows, by formula (2.9), that the matrices $\mathbf{V}_0$ and $\tilde{\mathbf{V}}_0$ are also real, and the matrices $\frac{\partial \mathbf{V}}{\partial \bar{\lambda}}(\bar{\lambda}_0, \mathbf{p}_0)$ and $\frac{\partial \tilde{\mathbf{V}}}{\partial \bar{\lambda}}(\bar{\lambda}_0, \mathbf{p}_0)$ are pure imaginary. Consequently, the function v_0 may be assumed to be real and v_1 may be assumed to be pure imaginary.

Since the functions u_0 and v_0 are defined, apart from arbitrary factors, and u_1 and v_1 apart from terms $\gamma_1 u_0$ and $\gamma_2 v_0$, respectively, where γ_1 and γ_2 are arbitrary constants, we can choose real functions u_0 and v_0 and pure imaginary functions u_1, v_1 that satisfy the normalization and orthogonality conditions

$$2i\omega_0(Mu_1, v_0) = 1, \quad 2i\omega_0(Mu_1, v_1) + (Mu_0, v_1) + (Mu_1, v_0) = 0 \tag{3.9}$$

We will investigate how the stability of system (3.1), (3.2) depends on linear perturbations of the parameter vector $\mathbf{p} = (\mathbf{k}, q)$

$$\mathbf{p}(\epsilon) = \mathbf{p}_0 + \epsilon\dot{\mathbf{p}}, \quad \epsilon \geq 0 \tag{3.10}$$

where the dot denotes a derivative with respect to the small parameter ϵ, evaluated at $\epsilon = 0$. In the case of the general position, the perturbed double eigenvalue is defined by a Newton–Puiseaux series (2.35). Substituting the operator L defined by Eq. (3.1) into Eqs (2.41) and (2.45) and using the normalization conditions (3.9), we obtain the coefficients λ_1 and λ_2:

$$\lambda_1^2 = -i\omega_0\langle \mathbf{f}, \dot{\mathbf{k}}\rangle - \tilde{f}\dot{q}, \quad 2\lambda_2 = -\langle \mathbf{f} - \omega_0\mathbf{h}, \dot{\mathbf{k}}\rangle - i\tilde{h}\dot{q} \tag{3.11}$$

where the vector $\dot{\mathbf{k}} = (\dot{k}_1, \dots, \dot{k}_{n-1})$, angular brackets denote the scalar product of real vectors in R^{n-1}, the components of the real vector $\mathbf{f}$ and real scalar $\tilde{f}$ are

$$f_r = \left(\frac{\partial D}{\partial k_r}u_0, v_0\right), \quad \tilde{f} = \left(\frac{\partial N}{\partial q}u_0, v_0\right) + \mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\frac{\partial \mathbf{U}_N}{\partial q}\mathbf{u}_0, \quad r = 1, \dots, n-1 \tag{3.12}$$

and the components of the real vector $\mathbf{h}$ and the real scalar $\tilde{h}$ are defined by

$$ih_r = \left(\frac{\partial D}{\partial k_r}u_1, v_0\right) + \left(\frac{\partial D}{\partial k_r}u_0, v_1\right), \quad r = 1, \dots, n-1 \tag{3.13}$$

$$i\tilde{h} = \left(\frac{\partial N}{\partial q}u_1, v_0\right) + \left(\frac{\partial N}{\partial q}u_0, v_1\right) + \mathbf{v}_1^*\tilde{\mathbf{V}}_0^*\frac{\partial \mathbf{U}_N}{\partial q}\mathbf{u}_0 + \mathbf{v}_0^*\tilde{\mathbf{V}}_0^*\frac{\partial \mathbf{U}_N}{\partial q}\mathbf{u}_1 + \mathbf{v}_0^*\left(\frac{\partial \tilde{\mathbf{V}}}{\partial \bar{\lambda}}\right)^*\frac{\partial \mathbf{U}_N}{\partial q}\mathbf{u}_0 \tag{3.14}$$

Thus, we obtain from Eqs (3.9)–(3.14)

$$\lambda = i\omega_0 \pm \sqrt{-i\omega_0\langle \mathbf{f}, \mathbf{k}\rangle - \tilde{f}(q - q_0)} - \frac{1}{2}(\langle \mathbf{f} - \omega_0\mathbf{h}, \mathbf{k}\rangle + i\tilde{h}(q - q_0)) + o(|\mathbf{p} - \mathbf{p}_0|) \tag{3.15}$$

Formula (3.15) describes the splitting of the double eigenvalue $i\omega_0$ due to variation of the parameters $\mathbf{k} = (k_1, \dots, k_{n-1})$ and q in the case when radicand does not vanish. If $\mathbf{k} = 0$, the double eigenvalue splits into two simple pure imaginary eigenvalues (stability), provided that $\tilde{f}(q - q_0) > 0$. We shall assume that $\tilde{f} < 0$. Then the system is stable for $q < q_0$ and unstable for $q > q_0$. The case $\tilde{f} = 0$ is degenerate and will not be considered here. For a sufficiently small variation of the parameters $\mathbf{k}$ and q, the double eigenvalue $i\omega_0$ generally splits into two simple complex eigenvalues, one of which has a positive real part (flutter). Nevertheless, if $\langle \mathbf{f}, \mathbf{k}\rangle = 0$ and $\langle \mathbf{h}, \mathbf{k}\rangle < 0$, the when $q < q_0$ the square root in Eq. (3.15) is pure imaginary, and for sufficiently small perturbations of the parameters the double eigenvalue $i\omega_0$ (and also $-i\omega_0$) splits into two simple eigenvalues with negative real parts (stability).

Asymptotic stability of system (3.1), (3.2) under perturbation (3.10) also depends on the behaviour of the simple pure imaginary eigenvalues $\pm i\omega_{0,s}$, $\omega_{0,s} > 0$. Choose real eigenfunctions $u_{0,s}$ and $v_{0,s}$ of the eigenvalue $i\omega_{0,s}$ satisfying the normalization conditions

$$2\omega_{0,s}(Mu_{0,s}, v_{0,s}) = 1 \tag{3.16}$$

By Eqs (2.23) and (2.26), the increments of the simple eigenvalues $\pm i\omega_{0,s}$ when the parameters are varied are defined by the equations

$$\lambda = \pm i\omega_{0,s} \mp i\tilde{g}_s(q - q_0) - \omega_{0,s}\langle \mathbf{g}_s, \mathbf{k}\rangle + o(|\mathbf{p} - \mathbf{p}_0|^2), \quad s = 1, 2, \ldots \tag{3.17}$$

The real quantity $\tilde{g}_s$ and the components of the real vector $\mathbf{g}_s$ have the form

$$\tilde{g}_s = \left(\frac{\partial N}{\partial q} u_{0,s}, v_{0,s}\right) + \mathbf{v}_0^* \tilde{\mathbf{V}}_0^* \frac{\partial \mathbf{U}_N}{\partial q} \mathbf{u}_{0,s}, \quad g_{s,r} = \left(\frac{\partial D}{\partial k_r} u_{0,s}, v_{0,s}\right), \quad r = 1, \ldots, n-1 \tag{3.18}$$

We have $\operatorname{Re}\lambda_s < 0$ if $\langle \mathbf{g}_s, \mathbf{k}\rangle > 0$.

It follows from relations (3.15) and (3.17) that system (3.1), (3.2) is asymptotically stable for sufficiently small linear variations of the parameters $\mathbf{k}$ and q, defined by Eq. (3.10), provided the following conditions hold

$$\langle \mathbf{f}, \mathbf{k}\rangle = 0, \quad q < q_0, \quad \langle \mathbf{h}, \mathbf{k}\rangle < 0, \quad \langle \mathbf{g}_s, \mathbf{k}\rangle > 0, \quad s = 1, 2, \ldots \tag{3.19}$$

These relations show that the set of directions leading from a point $\mathbf{p}_0$ to the asymptotic stability domain is of dimension $n - 1$ in the n-dimensional space of the parameters of the system $k_1, \ldots, k_{n-1}$, q. Thus, starting from the point $\mathbf{p}_0$, one can reach other points of the asymptotic stability domain only along curves that are tangent to the plane $\langle \mathbf{f}, \mathbf{k}\rangle = 0$ at $\mathbf{p}_0$. To get a more accurate picture of the geometry of the stability domain in the neighbourhood of the point $\mathbf{p}_0 = (0, \ldots, 0, q_0)$, let us consider a variation of the parameter vector along the smooth curve

$$\mathbf{p}(\epsilon) = \left\| \begin{matrix} 0 \\ q_0 \end{matrix} \right\| + \epsilon \left\| \begin{matrix} \dot{\mathbf{k}} \\ 0 \end{matrix} \right\| + \frac{\epsilon^2}{2} \left\| \begin{matrix} \ddot{\mathbf{k}} \\ \ddot{q} \end{matrix} \right\| + o(\epsilon^2) \tag{3.20}$$

assuming that

$$\langle \mathbf{f}, \dot{\mathbf{k}}\rangle = 0 \tag{3.21}$$

The curve (3.20), (3.21) is orthogonal to the q axis in the parameter space $\mathbf{k}$, q, since $\dot{q} \equiv 0$.

The coefficient λ_1 in expansion (2.35) defined by the first of equations (3.11) vanishes along the curve (3.20), (3.21). Consequently, a double eigenvalue $i\omega_0$ in this degenerate case splits into two simple eigenvalues that depend linearly on ϵ [34]

$$\lambda = i\omega_0 + \lambda_2\epsilon + o(\epsilon) \tag{3.22}$$

the coefficient λ_2 is a root of the quadratic equation which, for an operator L with boundary conditions given by (3.1), (3.2) and eigenfunctions and associated eigenfunctions satisfying Eqs (3.4), (3.5) and (3.7), (3.8), takes the form

$$\lambda_2^2 - \lambda_2\omega_0\langle \mathbf{h}, \dot{\mathbf{k}}\rangle + \left(\frac{1}{2}\tilde{f}\ddot{q} + \omega_0^2\langle \mathbf{G}\dot{\mathbf{k}}, \dot{\mathbf{k}}\rangle\right) + i\omega_0\left(\frac{1}{2}\langle \mathbf{f}, \ddot{\mathbf{k}}\rangle + \langle \mathbf{H}\dot{\mathbf{k}}, \dot{\mathbf{k}}\rangle\right) = 0 \tag{3.23}$$

The real vectors $\mathbf{f}$, $\mathbf{h}$ and the coefficients $\tilde{f}$, $\tilde{h}$, in Eq. (3.23) are defined by equalities (3.12)–(3.14), the real matrix $\mathbf{H}$ has components

$$H_{r\sigma} = \frac{1}{2}\left(\frac{\partial^2 D}{\partial k_r \partial k_\sigma} u_0, v_0\right), \quad r, \sigma = 1, \ldots, n-1 \tag{3.24}$$

and the real matrix $\mathbf{G}$ is defined by the expression

$$\langle \mathbf{G}\dot{\mathbf{k}}, \dot{\mathbf{k}}\rangle = \sum_{r=1}^{n-1} \dot{k}_r\left(\frac{\partial D}{\partial k_r}\hat{w}_2, v_0\right) \tag{3.25}$$

where $\hat{w}_2$ is a solution of the boundary-value problem

$$N(q_0)\hat{w}_2 - \omega_0^2 M\hat{w}_2 = \sum_{r=1}^{n-1} \dot{k}_r\frac{\partial D}{\partial k_r}u_0, \quad \mathbf{U}_N(q_0)\hat{\mathbf{w}}_2 = 0 \tag{3.26}$$

That a solution of this problem exists follows from the validity of a solvability condition equivalent to (3.21).

Noting the explicit equations (3.20), (3.21) of the curve $\mathbf{p}(\epsilon)$ and expression (3.22), we can write Eq. (3.23) in the form

$$(\lambda - i\omega_0)^2 - \omega_0\langle \mathbf{h}, \mathbf{k}\rangle(\lambda - i\omega_0) + \tilde{f}(q - q_0) + \omega_0^2\langle \mathbf{G}\mathbf{k}, \mathbf{k}\rangle + i\omega_0(\langle \mathbf{f}, \mathbf{k}\rangle + \langle \mathbf{H}\mathbf{k}, \mathbf{k}\rangle) = 0 \tag{3.27}$$

Equation (3.27) describes the collapse of the double eigenvalue $i\omega_0$ for a small perturbation of the parameters $\mathbf{k}$ and q. For a more detailed study of the process, we put $\lambda = \mathrm{Re}\lambda + i\mathrm{Im}\lambda$ in Eq. (3.27), separate real and imaginary parts, and, transforming, obtain

$$(\mathrm{Im}\lambda - \omega_0 + \mathrm{Re}\lambda + a/2)^2 - (\mathrm{Im}\lambda - \omega_0 - \mathrm{Re}\lambda - a/2)^2 = -2d \tag{3.28}$$

$$\left(\mathrm{Re}\lambda + \frac{a}{2}\right)^4 + \left(c - \frac{a^2}{4}\right)\left(\mathrm{Re}\lambda + \frac{a}{2}\right)^2 = \frac{d^2}{4} \tag{3.29}$$

$$(\mathrm{Im}\lambda - \omega_0)^4 - \left(c - \frac{a^2}{4}\right)(\mathrm{Im}\lambda - \omega_0)^2 = \frac{d^2}{4} \tag{3.30}$$

where

$$a = -\omega_0\langle \mathbf{h}, \mathbf{k}\rangle, \quad c = \tilde{f}(q - q_0) + \omega_0^2\langle \mathbf{G}\mathbf{k}, \mathbf{k}\rangle, \quad d = \omega_0(\langle \mathbf{f}, \mathbf{k}\rangle + \langle \mathbf{H}\mathbf{k}, \mathbf{k}\rangle) \tag{3.31}$$

We first consider the case in which the system is circulatory ($\mathbf{k} = 0$). Then, by Eq. (3.31), we have $a = 0$, $c = \tilde{f}(q - q_0)$, $d = 0$, and Eqs (3.29) and (3.30) become

$$q \le q_0\colon \mathrm{Re}\lambda = 0, \quad \mathrm{Im}\lambda = \omega_0 \pm \sqrt{\tilde{f}(q - q_0)} \tag{3.32}$$

$$q \ge q_0\colon \mathrm{Re}\lambda = \pm\sqrt{-\tilde{f}(q - q_0)}, \quad \mathrm{Im}\lambda = \omega_0 \tag{3.33}$$

Equations (3.32) and (3.33) show that, as the load parameter q is increased, the two pure imaginary eigenvalues move along the imaginary axis, meeting at $q = q_0$, forming a pair of double eigenvalues (flutter boundary), and then moving apart in directions perpendicular to the imaginary axis. Such behaviour of the eigenvalues, known as *strong interaction*, is typical of circulatory systems [22]. The trajectories of the eigenvalues of a circulatory system as the parameter q varies are shown in Figs 2 and 3 by the thin curves.

If $\mathbf{k} \ne 0$ and $d \ne 0$, the dissipative and gyroscopic forces disturb the strong interaction of the eigenvalues, displacing and splitting their trajectories, as shown in Figs 1 and 2 by the thick curves. This qualitative effect, known in the literature only from numerical solutions of particular mechanical problems [2, 12], is described *analytically* by Eqs (3.28)–(3.30).

In fact, for a fixed vector $\mathbf{k} \ne 0$, as the parameter q varies, the eigenvalues move in the complex plane along branches of a hyperbola (3.28) with two asymptotes, $\mathrm{Re}\lambda = -a/2$ and $\mathrm{Im}\lambda = \omega_0$, where a is defined by the first equation of (3.31). If $a > 0$, then one of the two eigenvalues is in the left half of the complex plane, whereas the other crosses the imaginary axis and enters the right half at $q = q_{cr}(\mathbf{k})$. Thus, the inequality $a > 0$ or, equivalently, $\langle \mathbf{h}, \mathbf{k}\rangle < 0$, is a necessary condition for asymptotic stability. Equations

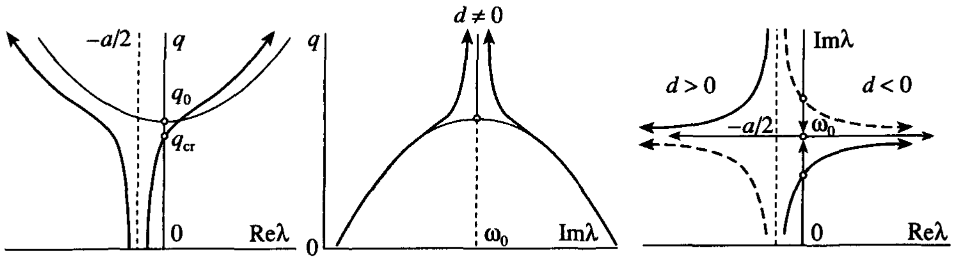

Fig. 2

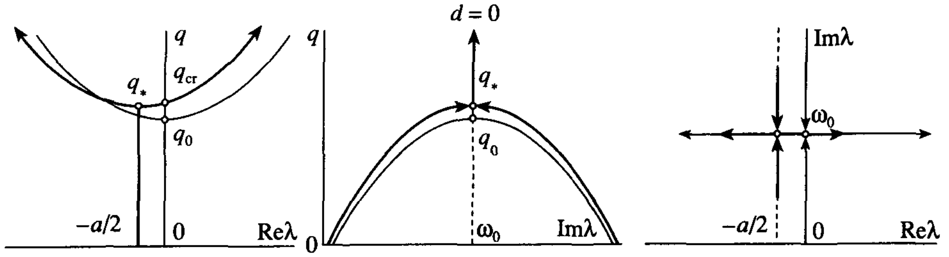

Fig. 3

(3.29) and (3.30) describe the real and imaginary parts of the eigenvalue λ as functions of the parameters q and $\mathbf{k}$. The functions Re $\lambda(q)$ and Im $\lambda(q)$ for $\mathbf{k} \neq 0$ are shown in Fig. 2 by the thick curves. The value of the parameter q at which one of the eigenvalues crosses the imaginary axis is obtained form Eq. (3.29) by assuming that Re $\lambda = 0$. This yields a relation $ca^2 = d^2$ which, given the explicit expressions (3.31) for the quantities a, c and d, becomes

$$q_{cr}(\mathbf{k}) = q_0 + \frac{(\langle \mathbf{f}, \mathbf{k} \rangle + \langle \mathbf{Hk}, \mathbf{k} \rangle)^2}{\tilde{f} \langle \mathbf{h}, \mathbf{k} \rangle^2} - \frac{\omega_0^2}{\tilde{f}} \langle \mathbf{Gk}, \mathbf{k} \rangle \tag{3.34}$$

Thus, both eigenvalues are situated in the left half of the complex plane if

$$q < q_{cr}(\mathbf{k}), \quad \langle \mathbf{h}, \mathbf{k} \rangle < 0 \tag{3.35}$$

The necessary and sufficient conditions (3.35) for all roots of the complex polynomial (3.27) to have negative real parts may also be obtained from Bilharz's criterion [38], which is an analogue for the Routh–Hurwitz criterion for complex polynomials.

Since by assumption $\tilde{f} < 0$, it follows from formula (3.34) that the critical load of a system with damping is such that $q_{cr}(\mathbf{k}) < q_0$ if $\langle \mathbf{Gk}, \mathbf{k} \rangle < 0$. But if $\langle \mathbf{Gk}, \mathbf{k} \rangle > 0$, there is a region in which, given a variation of the parameter vector $\mathbf{k}$ defined by the second inequality of (3.35) and the condition

$$(\langle \mathbf{f}, \mathbf{k} \rangle + \langle \mathbf{Hk}, \mathbf{k} \rangle)^2 - \omega_0^2 \langle \mathbf{Gk}, \mathbf{k} \rangle \langle \mathbf{h}, \mathbf{k} \rangle^2 < 0 \tag{3.36}$$

the critical load of the system with damping is such that $q_{cr}(\mathbf{k}) > q_0$.

Substituting Re $\lambda = 0$ into Eq. (3.28), we find an expression for the critical frequency

$$\omega_{cr}(\mathbf{k}) = \mathrm{Im}\lambda_{cr}(\mathbf{k}) = \omega_0 + \frac{\langle \mathbf{f}, \mathbf{k} \rangle + \langle \mathbf{Hk}, \mathbf{k} \rangle}{\langle \mathbf{h}, \mathbf{k} \rangle} \tag{3.37}$$

Hence it follows that the jump in critical frequency due to low damping $\mathbf{k} = \epsilon \tilde{\mathbf{k}}$ is

$$\Delta\omega \equiv \omega_0 - \lim_{\epsilon \to 0} \omega_{cr}(\epsilon \tilde{\mathbf{k}}) = -\frac{\langle \mathbf{f}, \tilde{\mathbf{k}} \rangle}{\langle \mathbf{h}, \tilde{\mathbf{k}} \rangle} \tag{3.38}$$

In the case when

$$d \equiv \omega_0(\langle \mathbf{f}, \mathbf{k}\rangle + \langle \mathbf{Hk}, \mathbf{k}\rangle) = 0$$

strong interaction of the eigenvalues is maintained by the introduction of small, velocity-dependent force ($\mathbf{k} \neq 0$). By formulae (3.29) and (3.30), which in this case are

$$q \le q_*: \operatorname{Re}\lambda = \omega_0 \frac{\langle \mathbf{h}, \mathbf{k}\rangle}{2}, \quad \operatorname{Im}\lambda = \omega_0 \pm \sqrt{\tilde{f}(q - q_*)} \tag{3.39}$$

$$q \ge q_*: \operatorname{Re}\lambda = \omega_0 \frac{\langle \mathbf{h}, \mathbf{k}\rangle}{2} \pm \sqrt{-\tilde{f}(q - q_*)}, \quad \operatorname{Im}\lambda = \omega_0 \tag{3.40}$$

complex eigenvalues λ with Re $\lambda = -a/2$ interact strongly at $q = q_*$, where

$$q_* = q_0 + \omega_0^2 \frac{\langle \mathbf{h}, \mathbf{k}\rangle^2 - 4\langle \mathbf{Gk}, \mathbf{k}\rangle}{4\tilde{f}} \tag{3.41}$$

When the parameter q is increased further, the double eigenvalue $\lambda_* = -a/2 + i\omega_0$ splits into two simple complex-conjugate eigenvalues (Fig. 3), one of which crosses the imaginary axis at a value of $q = q_{cr}(\mathbf{k})$ given by Eq. (3.34). This condition may be written as follows:

$$q_{cr}(\mathbf{k}) = q_0 - \frac{\omega_0^2}{\tilde{f}} \langle \mathbf{Gk}, \mathbf{k}\rangle \tag{3.42}$$

Thus, we arrive at the conclusion that in the case when $d = 0$ small damping forces only displace the picture of strong interaction of eigenvalues from the imaginary axis, as shown in Fig. 3 for $a > 0$. As in the previous case ($d \neq 0$), both eigenvalues are in the left half-plane if conditions (3.35) are satisfied. At the same time, as follows from formulae (3.42) and (3.39), (3.40), there is no jump of critical load or frequency. If in addition $\langle \mathbf{Gk}, \mathbf{k}\rangle > 0$ then, in accordance with Eq. (3.42), the critical load increases when there are small damping forces.

Let us confine ourselves to the case when

$$\{\mathbf{k}: \langle \mathbf{f}, \mathbf{k}\rangle = 0, \langle \mathbf{h}, \mathbf{k}\rangle < 0\} \subset \{\mathbf{k}: \langle \mathbf{g}_s, \mathbf{k}\rangle > 0, s = 1, 2, \ldots\} \tag{3.43}$$

implying that all simple eigenvalues $\pm i\omega_{0,s}$ for a small perturbation of the parameters q and $\mathbf{k}$, move into the left half of the complex plane, and therefore the stability of system, (3.1), (3.2) depends only on the collapse of double eigenvalues $\pm i\omega_0$. Thus, the surface $q_{cr}(k_1, \ldots, k_{n-1})$, which may be approximated by Eq. (3.34) under the restriction imposed by the second inequality of (3.35), is the boundary of the asymptotic stability domain (3.35) in a small neighbourhood of the point $\mathbf{p}_0 = (0, \ldots, 0, q_0)$.

The function $q_{cr}(\mathbf{k})$ defined by Eq. (3.34) is the sum of rational and polynomial parts. Both numerator and denominator of the rational part contain linear forms in the vector $\mathbf{k}$. Thus, the function $q_{cr}(\mathbf{k})$ has a singularity at the point $\mathbf{k} = 0$, and the critical load, as a function of $n - 1$ variables, does not have a limit as $\mathbf{k} = (k_1, \ldots, k_{n-1}) \to 0$. This fact was first established for the critical load of Zeigler's pendulum [13, 15] and has proved to be valid for arbitrary linear non-conservative systems with a finite number of degrees of freedom [20]. Nevertheless, the homogeneity of the numerator and denominator of the rational part of $q_{cr}(\mathbf{k})$ guarantees the existence of a limit $\lim_{\epsilon \to 0} q_{cr}(\epsilon\tilde{\mathbf{k}})$ for any direction $\tilde{\mathbf{k}}$ such that $\langle \mathbf{h}, \tilde{\mathbf{k}}\rangle \neq 0$. Substituting $\mathbf{k} = \epsilon\mathbf{k}$ in Eq. (3.34), we obtain an explicit expression approximating the *jump of critical load* due to small damping forces

$$\Delta q \equiv q_0 - \lim_{\epsilon \to 0} q_{cr}(\epsilon\tilde{\mathbf{k}}) = -\frac{1}{\tilde{f}} \frac{\langle \mathbf{f}, \tilde{\mathbf{k}}\rangle^2}{\langle \mathbf{h}, \tilde{\mathbf{k}}\rangle^2} \tag{3.44}$$

If $\langle \mathbf{f}, \tilde{\mathbf{k}}\rangle = 0$, there is no jump in the critical load ($\Delta q = 0$). For a two-dimensional vector $\mathbf{k} = (k_1, k_2)$, this condition yields the following ratio of the parameters k_1 and k_2

$$\frac{k_i}{k_j} = -\frac{f_j}{f_i}, \quad i, j = 1, 2 \tag{3.45}$$

for which small velocity-dependent forces do not reduce the critical load. The quantities f_1 and f_2 are defined by the first of equations (3.12). A strong dependence of the critical load on the ratio of the damping parameters was first observed by Bolotin [2, 4].

The function $q_{cr}(\mathbf{k})$, defined by Eq. (3.34) with the restriction (3.35), defines the boundary between the domains of asymptotic stability and flutter of system (3.1), (3.2) in the n-dimensional space of the parameters $\mathbf{k}$ and q. The level sets of the function (3.34) are the boundaries of the stability domain in the space of the parameters $\mathbf{k} = (k_1, \ldots, k_{n-1})$. The level set $q_{cr} = q_0$, where q_0 is the critical value of the parameter q in the unperturbed circulatory system (3.3), is given by the equation

$$\langle \mathbf{f}, \mathbf{k} \rangle = \pm\omega_0 \langle \mathbf{h}, \mathbf{k} \rangle \sqrt{\langle \mathbf{Gk}, \mathbf{k} \rangle} - \langle \mathbf{Hk}, \mathbf{k} \rangle \tag{3.46}$$

This equation has real solutions if $\langle \mathbf{Gk}, \mathbf{k} \rangle \geq 0$. In that case the set (3.46) bounds the domain of variation of the damping parameter vector (3.36) in which $q_{cr}(\mathbf{k}) > q_0$. If the matrix $\mathbf{G}$ is negative-definite, then $\langle \mathbf{Gk}, \mathbf{k} \rangle \leq 0$ and Eq. (3.46) has a unique real solution $\mathbf{k} = 0$, implying a drop in critical load (destabilization) for any small $\mathbf{k} \neq 0$.

Let us consider the case in which the damping parameter vector has two components, $\mathbf{k} = (k_1, k_2)$. Then the boundary of the stability domain, described by the function $q_{cr}(k_1, k_2)$, is a surface in the three-dimensional space of the parameter k_1, k_2 and q. To understand the structure of this surface, we shall find asymptotic formulae for the level curves of the function $q_{cr}(k_1, k_2)$ in the neighbourhood of the origin in the plane of the parameters k_1, k_2, for q_{cr} close to q_0.

We first approximate the level forces for $q_{cr} < q_0$ on the assumption that one of the parameters k_1, k_2 is a smooth function of the other. Substituting the expression

$$k_i = \beta_j k_j + o(k_j), \quad i, j = 1, 2$$

where β_j are unknown constants, into Eq. (3.34) and collecting terms with like powers of k_j, we obtain as a first approximation

$$k_i = -\frac{f_j \pm h_j \sqrt{\tilde{f}(q_{cr} - q_0)}}{f_i \pm h_i \sqrt{\tilde{f}(q_{cr} - q_0)}} k_j + o(k_j), \quad i, j = 1, 2 \tag{3.47}$$

Since by assumption $\tilde{f} < 0$ and $q_{cr} < q_0$, the square roots in Eq. (3.47) are real numbers. Consequently, for $q_{cr} < q_0$ the asymptotic stability domain in the k_1, k_2 plane is bounded in the first approximation by two straight lines intersecting at the origin, as shown in Fig. 4. Note that only the part of the surface $q_{cr}(k_1, k_2)$ that belongs to the half-space $\langle \mathbf{h}, \mathbf{k} \rangle < 0$ bounds the asymptotic stability domain.

It follows from Eq. (3.47) that, as q_{cr} increases, the angle between the curves bounding the asymptotic stability domain decreases, vanishing at $q_{cr} = q_0$. In that case the equations of the first approximation (3.47) define only the ratio of the parameters k_1 and k_2, as in (3.45). Substituting

$$k_i = -(f_j/f_i)k_j + \gamma_j k_j^2 + o(k_j^2)$$

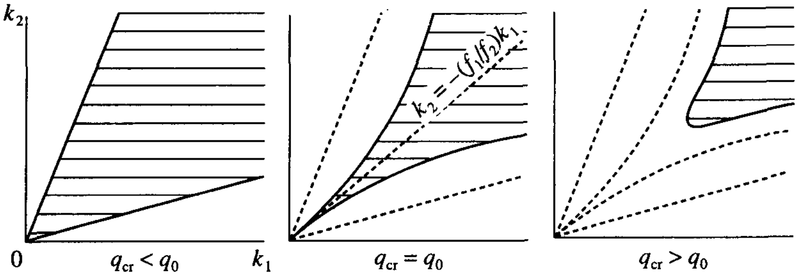

Fig. 4

where γ_j are unknown constants, into Eq. (3.46) and collecting terms with like power of k_j, we find a second approximation to the level curve $q_{cr} = q_0$

$$k_i = -\frac{f_j}{f_i}k_j - \frac{\mathbf{f}^T\mathbf{H}^\dagger\mathbf{f} \pm \omega_0(h_i f_j - h_j f_i)\sqrt{\mathbf{f}^T\mathbf{G}^\dagger\mathbf{f}}}{f_i^3}k_j^2 + o(k_j^2), \quad i, j = 1, 2 \tag{3.48}$$

$$\mathbf{H}^\dagger = \begin{Vmatrix} H_{22} & -H_{12} \\ -H_{21} & H_{11} \end{Vmatrix}, \quad \mathbf{G}^\dagger = \begin{Vmatrix} G_{22} & -G_{12} \\ -G_{21} & G_{11} \end{Vmatrix}$$

where H_{rs} and $G_{rs}(r, s = 1, 2)$ are the components of the matrices $\mathbf{H}$ and $\mathbf{G}$, defined by Eqs (3.24) and (3.25). Equation (3.48) describes two curves that touch at the origin of coordinates in the k_1, k_2 plane, forming a degenerate singularity known as a cuspidal point [15]. In the general case, the straight line $k_i = -(f_j/f_i)k_j$ is not always situated within the stability domain. But in the case when the matrix $\mathbf{D}(\mathbf{k})$ is a linear function of the parameters, it will always be in the asymptotic stability domain (Fig. 4), since the matrix $\mathbf{H}$, whose elements are the second derivatives of $\mathbf{D}(\mathbf{k})$ with respect to the parameters k_1 and k_2, vanishes.

To study the level curves for $q_{\text{cr}} > q_0$, we rewrite Eq. (3.34) as follows:

$$\langle\mathbf{f}, \mathbf{k}\rangle + \langle\mathbf{Hk}, \mathbf{k}\rangle = \pm\langle\mathbf{h}, \mathbf{k}\rangle\sqrt{\tilde{f}(q_{\text{cr}} - q_0) + \omega_0^2\langle\mathbf{Gk}, \mathbf{k}\rangle} \tag{3.49}$$

If $\langle\mathbf{Gk}, \mathbf{k}\rangle > 0$, real solutions of Eq. (3.49), describing level curves $q_{\text{cr}} > q_0$, exist provided that the radicand in (3.49) is positive, or, equivalently,

$$|\mathbf{k}| \equiv \sqrt{\langle\mathbf{k}, \mathbf{k}\rangle} > \sqrt{\frac{-\tilde{f}(q_{\text{cr}} - q_0)}{\omega_0^2\langle\mathbf{Ge}, \mathbf{e}\rangle}} > 0 \tag{3.50}$$

where $\mathbf{e} = \mathbf{k}/|\mathbf{k}|$. Condition (3.50) means that the level curves $q_{\text{cr}} > q_0$ do not pass through the origin. In addition, their distance from the origin depends on the right-hand side of inequality (3.50), as shown in Fig. 4.

Thus, having analysed the level curves of the function $q_{\text{cr}}(k_1, k_2)$, one can state that the boundary of the asymptotic stability domain, described by Eq. (3.34), in the neighbourhood of the point $(0, 0, q_0)$ in the space of the three parameters of the system, has the appearance shown in Fig. 5. This implies that the surface (3.34) has a singularity of "Whitney umbrella" type [39] at the point $(0, 0, q_0)$. The second condition of (3.35) cuts off half of the umbrella; the remaining part bounds the asymptotic stability domain, denoted by the letter S in Fig. 5. It is well-known that the Whitney umbrella is a singularity of general position of the boundary of the stable domains of three-parameter *finite-dimensional* non-conservative systems, corresponding to a double pure imaginary eigenvalue with Jordan chain of length 2 [16, 39]. In mechanical applications, this singularity was first found on the boundary of the stable domain of Ziegler's pendulum [14–16]. It was shown above that the Whitney umbrella is also a singularity of the boundary of the asymptotic stability domain in *distributed* non-conservative systems of type (3.1), (3.2) that depend on three parameters.

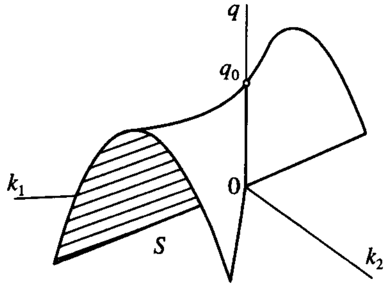

Fig. 5

4. EXAMPLE. THE STABILITY OF A VISCO-ELASTIC ROD

Let us return to problem (1.1), (1.2) – the problem of transverse vibrations in a viscous medium of a cantilever rod made of visco-elastic Kelvin–Voigt material, loaded at its free end by a tangential follower force q (Fig. 6). In dimensionless variables, the stability problem reduces to investigating the eigenvalue problem (1.3), (1.4) [7]. The matrix of the boundary conditions of problem (1.3), (1.4) is

$$\mathbf{U} = \left\| \begin{matrix} \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} \end{matrix} \right\| \tag{4.1}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{O}$ is the zero matrix of order 2×2. By formula (2.7), we have

$$\mathbf{L}(x) = \left\| \begin{matrix} -q\mathbf{J} & -(1+\eta\lambda)\mathbf{J} \\ -(1+\eta\lambda)\mathbf{J} & \mathbf{O} \end{matrix} \right\|, \quad \mathbf{J} = \left\| \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \right\| \tag{4.2}$$

Choosing the matrix $\tilde{\mathbf{U}}$ in the form

$$\tilde{\mathbf{U}} = \left\| \begin{matrix} \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} & \mathbf{O} & \mathbf{O} \end{matrix} \right\| \tag{4.3}$$

from formula (2.9) we obtain

$$\mathbf{V} = \left\| \begin{matrix} \mathbf{O} & \mathbf{O} & q\mathbf{J} & (1+\eta\bar{\lambda})\mathbf{J} \\ -(1+\eta\bar{\lambda})\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} \end{matrix} \right\|, \quad \tilde{\mathbf{V}} = \left\| \begin{matrix} q\mathbf{J} & (1+\eta\bar{\lambda})\mathbf{J} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & (-(1+\eta\bar{\lambda})\mathbf{J}) & \mathbf{O} \end{matrix} \right\| \tag{4.4}$$

By formulae (2.6) and (2.15), the adjoint of problem (1.3), (1.4) has the form

$$(1+\eta\bar{\lambda})v''''_{xxxx} + qv''_{xx} + (\bar{\lambda}^2 + \mu\bar{\lambda})v = 0 \tag{4.5}$$

$$(1+\eta\bar{\lambda})v'''_{xxx}(1) + qv'_x(1) = 0, \quad (1+\eta\bar{\lambda})v''_{xx}(1) + qv(1) = 0, \quad v'_x(0) = 0, \quad v(0) = 0 \tag{4.6}$$

A system without damping. When there is no damping $\mu = \eta = 0$, the spectrum of problem (1.3), (1.4) is defined by the characteristic equation [2, 36]

$$2\omega^2(1 + \operatorname{ch}(a)\cos(b)) + q(q + ab\operatorname{sh}(a)\sin(b)) = 0 \tag{4.7}$$

where

$$a = \sqrt{-q/2 + \sqrt{q^2/4 + \omega^2}}, \quad b = \sqrt{q/2 + \sqrt{q^2/4 + \omega^2}}, \quad \omega^2 = -\lambda^2 \tag{4.8}$$

It is well known that an elastic rod is stable magnitudes of the follower force in the interval $0 \le q < q_0$, where $q_0 = 20.05$ [21]. When $q = q_0$ the spectrum of problem (4.1), (4.2) is discrete. It consists of a pair of double eigenvalues $\pm i\omega_0(\omega_0 = 11.02)$ and simple eigenvalues $\pm i\omega_{0,s}$, $(s = 1, 2, \ldots)$, the sequence of simple frequencies being

$$\omega_{0,1} = 53.71, \quad \omega_{0,2} = 112.4, \quad \omega_{0,3} = 191.1, \ldots, \omega_{0,s\to\infty} = \pi^2 s^2 + O(s) \tag{4.9}$$

The asymptotic behaviour of the eigenvalues (4.9) was obtained in [7].

The double eigenvalue $i\omega_0$ has a Keldysh chain of length 2, consisting of the eigenfunction and associated eigenfunction, u_0, and u_1, which satisfy the equations with boundary conditions (1.5), (1.6) obtained from Eqs (1.3) and (1.4) by setting $\mu = \eta = 0$. The eigenfunction and associated eigenfunction of the complex conjugate eigenvalue $-i\omega_0$ satisfy equations and boundary conditions obtained from (4.5) and (4.6):

$$v''''_{0xxxx} + q_0 v''_{0xx} - \omega_0^2 v_0 = 0,\ v_0(0) = v'_{0x}(0) = 0,\ v''_{0xx}(1) + q_0 v_0(1) = \\ = v'''_{0xxx}(1) + q_0 v'_{0x}(1) = 0 \tag{4.10}$$

$$v''''_{1xxxx} + q_0 v''_{1xx} - \omega_0^2 v_1 = 2i\omega_0 v_0,\ v_1(0) = v'_{1x}(0) = 0,\ v''_{1xx}(1) + q_0 v_1(1) = \\ = v'''_{1xxx}(1) + q_0 v'_{1x}(1) = 0 \tag{4.11}$$

The eigenfunctions u_0 and v_0 are defined by Eqs (1.5) and (4.10) [36, 37]

$$u_0(x) = \mathrm{ch}(ax) - \cos(bx) + F(a\sin(bx) - b\,\mathrm{sh}(ax)) \tag{4.12}$$

$$v_0(x) = \mathrm{ch}(ax) - \cos(bx) + G(a\sin(bx) - b\,\mathrm{sh}(ax)) \tag{4.13}$$

where

$$F = \frac{a^2\mathrm{ch}(a) + b^2\cos(b)}{ab(a\,\mathrm{sh}(a) + b\sin(b))},\quad G = \frac{b^2\mathrm{ch}(a) + a^2\cos(b)}{b^3\mathrm{sh}(a) + a^3\sin(b)} \tag{4.14}$$

Solution of the boundary-value problem (1.6) yields the associated function u_1 [32]

$$u_1(x) = -2i\omega_0\frac{a\sin(bx) + b\,\mathrm{sh}(ax) + F(a^2\cos(bx) - b^2\mathrm{ch}(ax))}{2ab(a^2+b^2)}x - \\ -2i\omega_0\frac{A_1\mathrm{sh}(ax) - B_1\sin(bx)}{2ab(a^2+b^2)(a\,\mathrm{sh}(a) + b\sin(b))^2} \tag{4.15}$$

$$\begin{aligned} A_1 &= \frac{q}{a^2}(\sin(b)(b^2\cos(b) - a^2\mathrm{ch}(a)) + 2ab\cos(b)\mathrm{sh}(a)) + bC \\ B_1 &= \frac{q}{b^2}(\mathrm{sh}(a)(b^2\cos(b) - a^2\mathrm{ch}(a)) - 2ab\,\mathrm{ch}(a)\sin(b)) + aC \\ C &= (a^2+b^2)(1 + \mathrm{ch}(a)\cos(b)) \end{aligned} \tag{4.16}$$

The coefficient F is defined by the first equality of (4.14). The boundary-value problem (4.11) yields the associated eigenfunction v_1 [32]:

$$v_1(x) = 2i\omega_0\frac{a\sin(bx) + b\,\mathrm{sh}(ax) + G(a^2\cos(bx) - b^2\mathrm{ch}(ax))}{2ab(a^2+b^2)}x + \\ + 2i\omega_0\frac{A_2\mathrm{sh}(ax) - B_2\sin(bx)}{2ab(a^2+b^2)(b^3\mathrm{sh}(a) + a^3\sin(b))^2} \tag{4.17}$$

$$\begin{aligned} A_2 &= q\sin(b)(3a^2b^2\mathrm{ch}(a) + a^4\cos(b)) - 2qab^3\mathrm{sh}(a)\cos(b) + b(b^2a^2C + q^2(a^2+b^2)) \\ B_2 &= 2qba^3\sin(b)\mathrm{ch}(a) - q\,\mathrm{sh}(a)(3a^2b^2\cos(b) + b^4\mathrm{ch}(a)) + a(b^2a^2C + q^2(a^2+b^2)) \end{aligned} \tag{4.18}$$

The coefficient G is defined by the second equality of (4.14).

The eigenfunctions and associated functions just obtained, defined on the boundary between the domains of stability and flutter of the elastic rod at the point $\mu = \eta = 0, q = q_0$, may be used to compute an approximation to the boundary of the asymptotic stability domain of the rod when $\mu \neq 0, \eta \neq 0$.

A rod with damping. We will now considered a visco-elastic rod vibrating in a viscous medium. Let us investigate the effect of small internal damping ($\eta \neq 0$) and external damping ($\mu \neq 0$) on the simple eigenvalues $\pm i\omega_{0,s}$ ($s = 1, 2, \ldots$). The behaviour of the simple eigenvalues as the parameters are varied is described by formulae (3.17); the increment in the real part of the perturbed eigenvalues $\pm i\omega_{0,s}$ is defined by the vectors $\mathbf{g}_s$ computed by formula (3.18). For problem (1.3), (1.4), these vectors take the form

$$\mathbf{g}_s = \frac{1}{2\omega_{0,s}}\left(\frac{(u_{0,sxxxx}'''', v_{0,s})}{(u_{0,s}, v_{0,s})}, 1\right) \tag{4.19}$$

The eigenfunctions $u_{0,s}$ and $v_{0,s}$ of the simple eigenvalues $\pm i\omega_{0,s}$ are defined by Eqs (4.12) and (4.13). As $s \to \infty$, the eigenfrequencies have the asymptotic behaviour defined by (4.9), and the following asymptotic expansions hold for the corresponding eigenfunctions [7]

$$u_{0,s} = \sin(s\pi x) + O(s^{-1}), \quad v_{0,s} = \sin(s\pi x) + O(s^{-1}) \tag{4.20}$$

Using the eigenfrequencies (4.9) and eigenfunctions (4.12), (4.13), (4.20), we find the vectors $\mathbf{g}_s$ from formula (4.19)

$$\mathbf{g}_1 = (35.44, 0.009), \quad \mathbf{g}_2 = (65.03, 0.004), \quad \mathbf{g}_3 = (104.5, 0.003), \ldots \tag{4.21}$$

$$\mathbf{g}_s = \frac{1}{2}(s^2\pi^2 + o(s^2), s^{-2}\pi^{-2} + o(s^{-2})), \quad s \to \infty \tag{4.22}$$

By conditions (3.19), all the simple eigenvalues are displaced under the action of small damping into the left half-plane, if all the scalar products $\langle \mathbf{g}_s, \mathbf{k}\rangle$ ($s = 1, 2, \ldots$) are positive, where $\mathbf{k} = (\eta, \mu)$. It follows from relations (4.21) and (4.22) that this infinite set of inequalities is equivalent to just two conditions

$$\eta > 0, \quad \mu > -3807\eta \tag{4.23}$$

corresponding to the limit $\lim_{s\to\infty}\mathbf{g}_s$ and the vector $\mathbf{g}_1$, respectively. Incidentally, the behaviour of the simple eigenvalues of the system with low damping and their effect on stability has never been analysed before.

We will now find the stability conditions derived from the information about the collapse of the double eigenvalues $\pm i\omega_0$. We first observe that the eigenfunctions of a double eigenvalue (4.12), (4.13) satisfy an orthogonality condition (2.34), which for problem (1.3), (1.4) has the following form

$$(u_0, v_0) = 0 \tag{4.24}$$

Substituting the differential operator and matrices of the boundary conditions defined by Eqs (1.3), (4.1), (4.3) and (4.4) into Eqs (3.12)–(3.14) and taking condition (4.24) into consideration, we find

$$\tilde{f} = \frac{(u_{0xx}'', v_0)}{2i\omega_0(u_1, v_0)}, \quad \mathbf{f}_1 = \left(\frac{(u_{0xxxx}'''', v_0), 0}{2i\omega_0(u_1, v_0)}\right), \quad \mathbf{h} = -\left(\frac{(u_{0xxxx}'''', \bar{v}_1) + (u_{1xxxx}'''', v_0)}{2\omega_0(u_1, v_0)}, \frac{1}{\omega_0}\right) \tag{4.25}$$

It follows from the last formula of (4.25) and from expressions (3.28) and (3.31) that the contribution of a small external damping with coefficient μ to the increment of the real part of the perturbed double eigenvalue is $-\mu/2$. This result is not new [4, 12].

Substituting the eigenfunctions and associated functions (4.12), (4.13) and (4.15), (4.17), subject to conditions (3.9) and evaluated at $q = q_0$ and $\omega = \omega_0$, into formulae (4.25), we obtain

$$\tilde{f} = -4.730, \quad \mathbf{f} = (94.84, 0), \quad \mathbf{h} = -(14.34, 0.091) \tag{4.26}$$

The matrix $\mathbf{H} \equiv 0$, since the operator defined by Eq. (1.3) depends linearly on the parameters. To evaluate the matrix $\mathbf{G}$ with the help of Eq. (3.25), one has to solve the boundary-value problem (3.26), which now has the form

$$\hat{w}_{2xxxx}'''' + q_0\hat{w}_{2xx}'' - \omega_0^2\hat{w}_2 = \dot{\mu}u_0, \quad \hat{w}_2(0) = \hat{w}_{2x}'(0) = 0, \quad \hat{w}_{2xx}''(1) = \hat{w}_{2xxx}'''(1) = 0 \tag{4.27}$$

Comparing the boundary-value problems (4.27) and (1.6), we find that $\hat{w}_2 = -\dot{\mu}u_1/(2i\omega_0)$ (the function u_1 is defined by Eq. (4.15)). Taking this expression into consideration in (3.25), we obtain

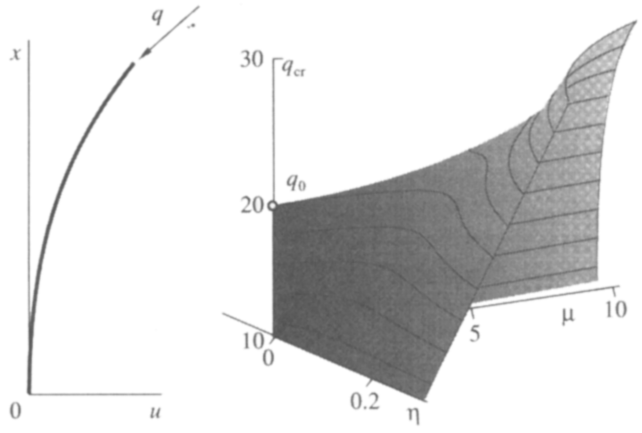

Fig. 6

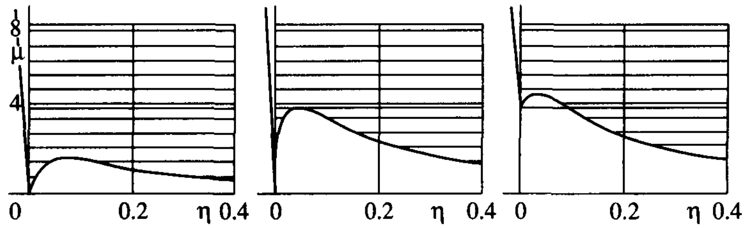

Fig. 7

$$\mathbf{G} = \frac{1}{8\omega_0^2(u_1, v_0)} \left\| \begin{array}{cc} 0 & (u''''_{1xxxx}, v_0) \\ (u''''_{1xxxx}, v_0) & 2(u_1, v_0) \end{array} \right\| \tag{4.28}$$

Substituting the eigenfunctions and associated eigenfunctions into (4.28) we obtain

$$\mathbf{G} = \left\| \begin{array}{cc} 0 & 0.247 \\ 0.247 & 0.002 \end{array} \right\| \tag{4.29}$$

Using the quantities (4.26) and (4.29), we use formula (3.34) to find an approximation to the critical load as a function of the dissipation parameters

$$q_{cr}(\eta, \mu) = q_0 - \frac{1902\eta^2}{(14.34\eta + 0.091\mu)^2} + 12.68\eta\mu + 0.053\mu^2 \tag{4.30}$$

The necessary condition for stability $\langle \mathbf{h}, \mathbf{k} \rangle < 0$ now becomes

$$\mu > -158.0\eta \tag{4.31}$$

A analytical formula (4.30) is the new expression for the critical load for the system with external and internal damping. The critical load function (4.30), when condition (4.31) is satisfied, is illustrated in Fig. 6.

Combining the stability conditions obtained by investigating the behaviour of simple and double eigenvalues, we find that our visco-elastic rod in a viscous medium is asymptotically stable in the neighbourhood of the point

$$\eta = 0, \quad \mu = 0, \quad q = q_0$$

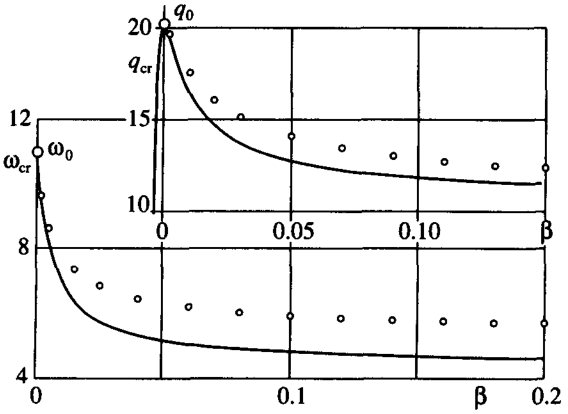

Fig. 8

provided that the following three conditions hold

$$q < q_{cr}(\eta, \mu), \quad \eta > 0, \quad \mu > -158.0\eta \tag{4.32}$$

The quantity $q_{cr}(\eta, \mu)$ is defined by formula (4.30).

Since both coefficients of damping were assumed to be non-negative, the last two conditions of (4.32) are automatically satisfied.

Figure 7 illustrates sections of the asymptotic stability domain (4.32) in the plane of the damping parameters η, μ, for different q values.

It is clear from Figs 6 and 7 that there is an asymptotic stability domain in the space of the three parameters with a singularity at the point $\eta = 0$, $\mu = 0$, $q = q_0$. This domain is strongly stretched along the vertical axis, corresponding to the coefficient of external damping μ. In addition, it follows from an analysis of the level curves of the boundary of the stability domain that a domain of variation for the damping parameters exists in which $q_{cr}(\eta, \mu) > q_0$. This leads to the following conclusion with regard to the *stabilizing effect* of internal and external damping on a visco-elastic rod: given any small coefficient of internal damping η, a small value of the coefficient of external damping μ exist for which $q_{cr}(\eta, \mu) > 0$ and the system with damping is asymptotically stable. This effect was not observed in earlier work on distributed systems.

Expressions approximating the jumps of critical load and frequency may be obtained from Eqs (3.38) and (3.44) by substituting the quantities (4.26) into them

$$\Delta q = \frac{1902\beta^2}{(14.34\beta + 0.091)^2}, \quad \Delta\omega = \frac{94.84\beta}{14.34\beta + 0.091}, \quad \beta = \frac{\eta}{\mu} \tag{4.33}$$

The solid curves in Fig. 8 represent the critical load and frequency as functions of the ratio β of the coefficients of internal and external damping, as evaluated from formulae (4.33); the results are in good agreement with previous, numerically evaluated, data [7], shown in Fig. 8 by small circles. The accuracy of the approximations (4.33) is best in the neighbourhood of $\beta = 0$. Nevertheless, the limits of the critical load and frequency at $\mu = 0$ as $\eta \to 0$ are $q_{cr} = 10.80$ and $\omega_{cr} = 4.40$, respectively, which are close to the values $q_{cr} = 10.94$ and $\omega_{cr} = 5.40$ obtained numerically in [7].

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